Coding and Information Theory Chapter 7: Linear Codes - E

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Chapter 7: Linear Codes

1. Matrix Description of Linear Codes
2. Equivalence of Linear Codes
3. Minimum Distance of Linear Codes
4. The Hamming Codes
5. The Golay Codes
6. The Standard Array
7. Syndrome Decoding

Quick Review of Last Lecture

- Minimum Distance of Linear Codes
  - Corollary 7.31: A linear $[n, k]$-code is t-error-correcting if and only if every set of $2t$ columns of its parity-check matrix are linearly independent
- The Hamming Codes H_n
  - $n = 2^C - 1$
  - Construct Hamming codes H_n
  - Nearest neighbor decoding with H_n
- The Standard Array
  - Construct the standard array of a linear code C
  - Decoding rule with using the standard array

7.7 Syndrome Decoding

- If H is a parity-check matrix for a linear code $C \subseteq V$ then the syndrome of a vector $v \in V$ is the vector

$$\mathbf{s} = \mathbf{v}H^{\mathrm{T}} \in F^{n-k} \qquad (7.8)$$

- Lemma 7.42
  - Let C be a linear code, with parity-check matrix H, and let $v, v' \in V$ have syndromes s, s'. Then v and v' lie in the same coset of C if and only if $s = s'$.
- Proof of Lemma 7.42

$$\begin{aligned} \mathbf{v} + \mathcal{C} = \mathbf{v}' + \mathcal{C} &\iff \mathbf{v} - \mathbf{v}' \in \mathcal{C} \\ &\iff (\mathbf{v} - \mathbf{v}')H^{\mathrm{T}} = \mathbf{0} \qquad \text{(by Lemma 7.10)} \\ &\iff \mathbf{v}H^{\mathrm{T}} = \mathbf{v}'H^{\mathrm{T}} \\ &\iff \mathbf{s} = \mathbf{s}'. \end{aligned}$$

Syndrome Table

- Lemma 7.42 shows that
  - A vector $\boldsymbol{v} \in V$ lies in the i-th row of the standard array if and only if it has the same syndrome as $\boldsymbol{v_i}$, that is,

$$\boldsymbol{v}H^T = \boldsymbol{v_i}H^T.$$

- A syndrome table can be created with each row having a coset leader $\boldsymbol{v_i}$ and its syndrome $\boldsymbol{s_i}$ $(= \boldsymbol{v_i}H^T)$.

A Syndrome Table Example 7.43

- Let C be the binary repetition code R_4, with standard array as given in Example 7.39, so the coset leaders $\boldsymbol{v_i}$ are the words in its first column.
- Apply the parity-check matrix given in Example 7.11.

$$H = \begin{pmatrix} 1 & & & -1 \\ & 1 & & -1 \\ & & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & & & 1 \\ & 1 & & 1 \\ & & 1 & 1 \end{pmatrix}$$

- Compute syndrome $\boldsymbol{s_i}$ for each $\boldsymbol{v_i}$.

$\mathbf{v}_i$	$\mathbf{s}_i$
0000	000
1000	100
0100	010
0010	001
0001	111
1100	110
1010	101
1001	011

Syndrome Decoding

The syndrome decoding proceeds as follows

- Given any received $\boldsymbol{v}$, compute its syndrome $\boldsymbol{s} = \boldsymbol{v}H^T$.
- Find $\boldsymbol{s}$ in the second column of the syndrome table, say $\boldsymbol{s} = \boldsymbol{s_i}$, the i-th entry.
- If $\boldsymbol{v_i}$ is the coset leader corresponding to $\boldsymbol{s_i}$ in the table, Then decode v as $u_j = v - v_i$. I.e.

$$\Delta(\mathbf{v}) = \mathbf{u}_j = \mathbf{v} - \mathbf{v}_i\,, \quad \text{where} \quad \mathbf{v}H^{\mathrm{T}} = \mathbf{s}_i$$

A Syndrome Decoding Example 7.44

$$\Delta(\mathbf{v}) = \mathbf{u}_j = \mathbf{v} - \mathbf{v}_i\,, \quad \text{where} \quad \mathbf{v}H^{\mathrm{T}} = \mathbf{s}_i$$

As in Example 7.43.

- v = 1101 is received.
- its syndrome $\boldsymbol{s} = \boldsymbol{v}H^T$ = 001.
- This is s_4 in the syndrome table, so we decode v as

$$\Delta(\mathbf{v}) = \mathbf{v} - \mathbf{v}_4 = 1101 - 0010 = 1111$$

$\mathbf{v}_i$	$\mathbf{s}_i$
0000	000
1000	100
0100	010
0010	001
0001	111
1100	110
1010	101
1001	011