Coding and Information Theory
Chapter 7:
Linear Codes - C

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Chapter 7: Linear Codes

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Quick Review of Last Lecture

- Matrix Description of Linear Codes
  - Linear code $C \subseteq V$ = F^n and let dim(C) = k
  - Dual Code D of C: dim(D) = $n - k$
  - Orthogonal Code $C^\perp$ of C: $D = C^\perp$ and $C = D^\perp$
  - Examples:
    - $C = C^\perp$
    - ${R_n}^\perp = P_n$ and ${P_n}^\perp = R_n$
    - The code $H_7^\perp$ is a linear [7, 3]-code over F_2
  - The conditions for H to be a parity-check matrix for C

7.2 Equivalence of Linear Codes

- The elementary row operations of matrix consist of
  - permuting rows,
  - multiplying a row by a non-zero constant, and
  - replacing a row r_i with $r_i + ar_j$ where $j \neq i$ and $a \neq 0$.
- Two linear codes C_1 and C_2 are **equivalent** if they have generator matrices G_1 and G_2 which differ only by elementary row operations and permutations of columns.
  - Elementary row operations on generator G may change the basis for C, but they do not change the subspace C.
  - Permutations of columns of G may change C, but the new code will differ from C only in the order of symbols within code-words.

Equivalence of Linear Codes (Cont.)

- By systematically using elementary row operations and column permutations, one can convert any generator matrix into the form

$$G = (I_k \mid P) = \begin{pmatrix} 1 & & & & * & * & \dots & * \\ & 1 & & & * & * & \dots & * \\ & & \ddots & & \vdots & \vdots & & \vdots \\ & & & 1 & * & * & \dots & * \end{pmatrix} \quad (7.2)$$

- We then say that G (or C) is in systematic form.
  - In this case, each $\boldsymbol{a} = a_1 \dots a_k \in F^k$ is encoded as

$$\mathbf{u} = \mathbf{a}G = a_1 \dots a_k a_{k+1} \dots a_n$$

  - where $a_1 \dots a_k$ are information digits and $a_{k+1} \dots a_n = \boldsymbol{a}P$ is a block of n - k check digits.

Two Examples

- Example 7.18
  - The generator matrices G for the codes R_n and P_n are in systematic form.

$$G = \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix}$$

$$G = \begin{pmatrix} 1 & & & & -1 \\ & 1 & & & -1 \\ & & \ddots & & \vdots \\ & & & 1 & -1 \end{pmatrix}$$

- Example 7.19.
  - The generator matrix G for H_7, is not in systematic form.
  - But, it can be transformed into systematic form.

$$G_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \quad G_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Equivalence of Linear Codes (Cont.)

- If we have a generator matrix $G = (I_k|P)$ in systematic form for a linear code C, then we can find a parity-check matrix for C.

$$H = (-P^{\mathrm{T}} \mid I_{n-k}) \qquad (7.3)$$

  - This is the systematic form for a parity-check matrix
- Prove this by using Lemma 7.17
  - H has n - k rows and n columns
  - Its rows are independent
  - $GH^{\mathrm{T}} = I_k(-P) + PI_{n-k} = -P + P = 0\,.$

Parity-check matrix in systematic form

$$G = (I_k | P) \qquad H = (-P^T | I_{n-k})$$

- Example 7.20: For the code R_n

$k = 1$

$G = (1,1, \dots, 1)_{1 \times n}$

$P = (1, \dots, 1)_{1 \times (n-1)}$

$$H = (-P^T | I_{n-1}) = \begin{pmatrix} -1 & 1 & & & \\ -1 & & 1 & & \\ \vdots & & & \ddots & \\ -1 & & & & 1 \end{pmatrix}_{(n-1) \times n}$$

- Example 7.21: For the code P_n

$k = n - 1$

$$G = \begin{pmatrix} 1 & & & & -1 \\ & 1 & & & -1 \\ & & \ddots & & \vdots \\ & & & 1 & -1 \end{pmatrix}_{(n-1) \times n}$$

$P^T = (-1, \dots, -1)_{1 \times (n-1)}$

$H = (1,1, \dots, 1)_{1 \times n}$

Parity-check matrix in systematic form

$G = (I_k|P)$ $H = (-P^T|I_{n-k})$

- Example 7.22: for the code H_7

$k = 4$

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$H = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

The Singleton Bound

- Exercise 6.18

  Prove the Singleton bound: if a code C over F_q has length n, minimum distance d, and M code-words, then

  $$\log_q M \leq n - d + 1.$$

Deleting $d-1$ symbols from each code-words in C, then C still has M distinct words of length $n-d+1$ over $F_{q\,.}$

There are at most q^{n-d+1} words of length $n-d+1$ over F_q, so $M \leq q^{n-d+1}$

- Theorem 7.23

  If C is a linear code of length n, dimension k, and minimum distance d, then

  $$d \leq 1 + n - k.$$

  - Two proofs

$M = q^k$

Generator of C in systematic form $G = (I_k|P)$

Weight of each row vector of $G \leq 1 + n - k$

So, $d \leq 1 + n - k$

The Singleton Bound

$$d \leq 1 + n - k.$$

- Example 7.24
  - The Singleton bound is attained by R_n
    - with k = 1 and d = n,
  - The Singleton bound is also attained by P_n
    - with k = n - 1 and d = 2;
  - But, not by H_7,
    - with d = 3 and 1 + n - k = 4,
- Corollary 7.25
  - A t-error-correcting linear [n, k]-code requires at least 2t check digits.
- Example 7.26
  - The linear codes R_3 and H_7 both have t = 1; the number of check digits is n - k = 2 or 3 respectively.

7.3 Minimum Distance of Linear Codes

- Theorem 7.27
  - Let C be a linear code of minimum distance d, and let H be a parity-check matrix for C. Then d is the minimum number of linearly dependent columns of H.
- Proof
  - Let $v = v_1 v_2 \dots v_n \in V$ and $H = (c_1 c_2 \dots c_n)$
  - $v \in C \Leftrightarrow vH^T = 0 \Leftrightarrow v_1 c_1 + v_2 c_2 + \cdots + v_n c_n = 0$
  - weight of v in C

    = number of non-zero v_i's

    = number of c_i's that are linearly dependent
  - d = minimum weight of code-words in C

    = the minimum number of c_i's that are linearly dependent

    = the minimum number of linearly dependent columns of H

Minimum Distance of Linear Codes (Cont.)

- Meaning of linearly dependent of columns of H
  - One column $\boldsymbol{c_i}$ linearly dependent, then $\boldsymbol{c_i} = \boldsymbol{0}$
  - Two columns $\boldsymbol{c_i}$ and $\boldsymbol{c_j}$ linearly dependent, then $\boldsymbol{c_i}$ is multiple of $\boldsymbol{c_j}$ (or $\boldsymbol{c_j}$ is multiple of $\boldsymbol{c_i}$).
  - So, $d \geq 3$ if and only if the columns of H are non-zero and none is a multiple of any other.

- Example 7.28
  - The parity-check matrix H = (1 1 … 1) for P_n has its columns non-zero and equal , so P_n has minimum distance d = 2.

Minimum Distance of Linear Codes (Cont.)

- Example 7.29

  In the parity-check matrix H for R_n, any set of n - 1 columns are linearly independent, while $c_1 + \cdots + c_n = 0$. So d = n.

$$H = \begin{pmatrix} 1 & & & & -1 \\ & 1 & & & -1 \\ & & \ddots & & \vdots \\ & & & 1 & -1 \end{pmatrix}$$

- Example 7.30

  Now, look at the parity-check matrix H for H_7

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$