# Coding and Information Theory Overview
# Chapter 1: Source Coding

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**The second Lecture of Chapter 1**

# Quick Review (1)

- The source alphabet of $S = \{s_1, s_2, \ldots, s_q\}$
- Probability distribution $P = (p_1, p_2, \ldots, p_q)$
- Code alphabet $T = \{t_1, \ldots, t_r\}$
- Code word: a sequence of symbols from $T$
- Encode source $s$ = $X_1 X_2 X_3$ ..., where $X_n$ = $s_i$
- Source code (simply code) $C$ = $\{w_1, w_2, \ldots, w_q\}$
- The average word-length of $C$ is

$$L(C) = \sum_{i=1}^{q} p_i l_i . \quad where\ l_i = |w_i|$$

# Quick Review (2)

- A uniquely decodable code $C$
- Compute $C_n$ and $C_\infty$

    Algorithm to compute C1, C2, …, Cn_1, Cn

        C0 = C

        For each code-word cw in C

            For each code cw_1 in Cn_1

                If cw_1 is prefix of cw, then add cw/cw_1 in Cn

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- The Sardinas-Patterson Theorem (Theorem 1.10)
  - A code $C$ (finite) is uniquely decodable if and only if $C \cap C_\infty = \emptyset$

# Example 1.12

- Let $C$ be the ternary code {01, 1, 2, 210}.
    - C1 = {10}
    - C2 = {0}
    - C3 = {1}
    - Now $1 \in C \cap C_{\infty}$. So $C$ is not uniquely decodable

- 10 $\in$ C1 because **2.10 = 210,** with 2 $\in$ C and 210 $\in$ C
- 0 $\in$ C2 because **1.0 = 10,** with 1 $\in$ C and 10 $\in$ C1
- 1 $\in$ C3 because **0.1 = 01,** with 0 $\in$ C2 and 01 $\in$ C

# Example 1.12 (Cont.)

- Let $C$ be the ternary code {01, 1, 2, 210}.
  - C1 = {10} because **2.10 = 210,** with 2 $\in$ C and 210 $\in$ C
  - C2 = {0} because **1.0 = 10,** with 1 $\in$ C and 10 $\in$ C1
  - C3 = {1} because **0.1 = 01,** with 0 $\in$ C2 and 01 $\in$ C
  - Now $1 \in C \cap C_\infty$. So *C* is not uniquely decodable
- Can you find an example of non-unique decodability?

# Proof of Theorem 1.10

## A code $C$ (finite) is uniquely decodable if and only if $C \cap C_{\infty} = \emptyset$

- ($\Rightarrow$) Suppose that $C \cap C_{\infty} \neq \emptyset$, want to prove C is not u.d.

- ($\Leftarrow$) Suppose that the code $C$ is not u.d., want to prove $C \cap C_{\infty} \neq \emptyset$

# 1.3 Instantaneous Codes

- Example 1.14
  - Consider the binary code $C$ given by
    $$s_1 \mapsto 0,\ s_2 \mapsto 01,\ s_3 \mapsto 11.$$
  - We have $\mathcal{C}_1 = \mathcal{C}_2 = \cdots = \{1\}$, so $\mathcal{C}_\infty = \{1\}$;
  - Thus $\mathcal{C} \cap \mathcal{C}_\infty = \emptyset$, so $C$ is uniquely decodable
  - Consider a finite message $t$ = 0111 . ...
  - We can not decode until we know how many 1's.
  - We say that $C$ is not **instantaneous**.

# Instantaneous Codes (cont.)

- Example 1.16
  - Consider the binary code $D$ given by

    $$s_1 \mapsto 0,\ s_2 \mapsto 10,\ s_3 \mapsto 11,$$
  - the reverse of the code $C$ in Example 1.14
  - this is uniquely decodable
  - It is also instantaneous
- Formal definition
  - A code $C$ is instantaneous if, for each sequence of code-words $w_{i_1}w_{i_2}, \ldots w_{i_n}$, every code-sequence beginning $t = w_{i_1}w_{i_2}, \ldots w_{i_n} \ldots$ is decoded uniquely as $s = s_{i_1}s_{i_2} \ldots s_{i_n} \ldots$, no matter what the subsequent symbols in $t$ are.

# Prefix Code

- A code $C$ is a prefix code if no code-word $w_i$ is a prefix (initial segment) of any code-word $w_j$ $(i \neq j)$; equivalently, $w_j \neq w_i w$ for any $w \in T^*$,
  - That is, $C_1 = \emptyset$
- **Theorem 1.17**
  - A code $C$ is instantaneous if and only if it is a prefix code.

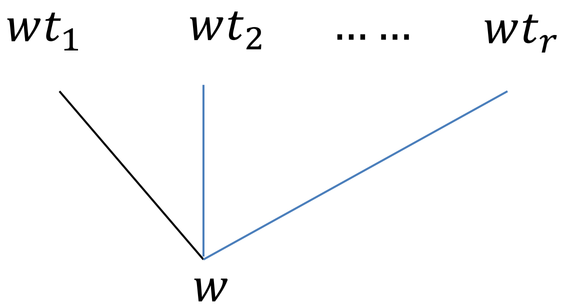
(⇒) If not prefix, then not instantaneous

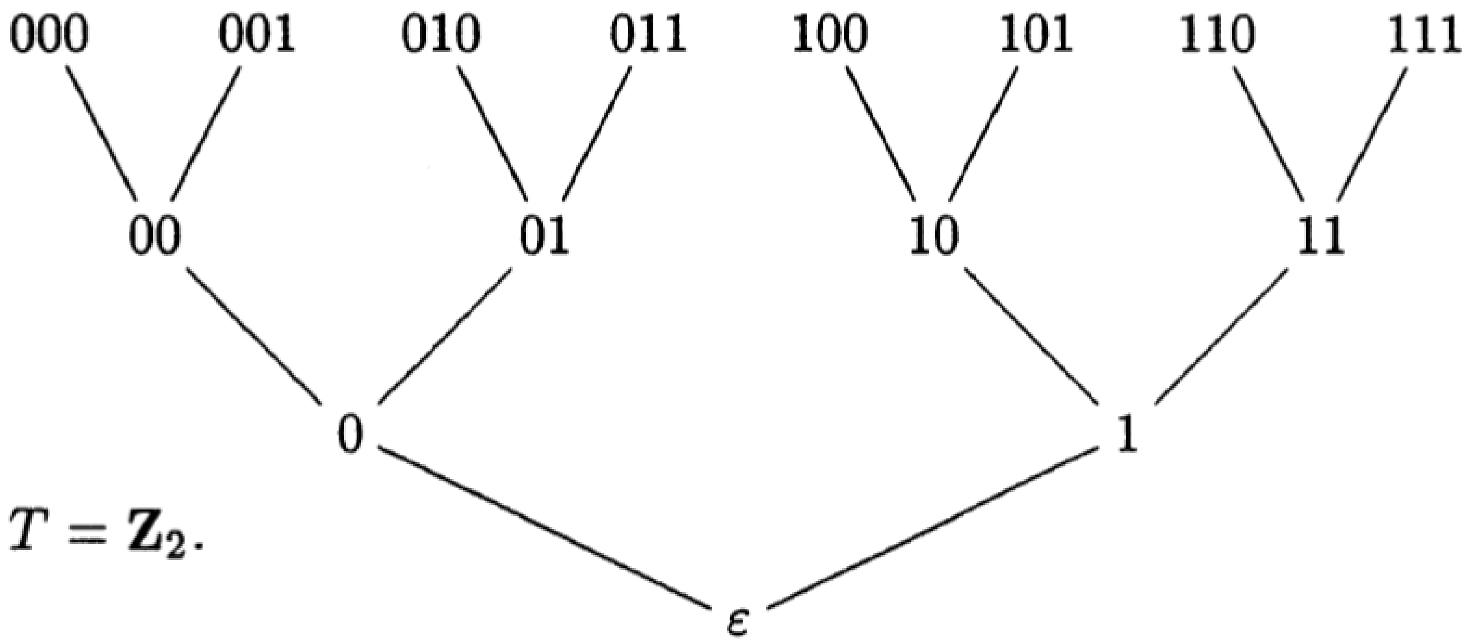
(⇐) If prefix, then instantaneous

- Note: A code $C$ is instantaneous, if
  - $t = w_{i_1} w_{i_2}, \ldots w_{i_n}$ … is decoded uniquely as
  - $s = s_{i_1} s_{i_2} \ldots s_{i_n} \ldots,$

# 1.4 Constructing Instantaneous Codes

- $w \in T^*$
- $T = \{t_1, t_2, .., t_r\}$

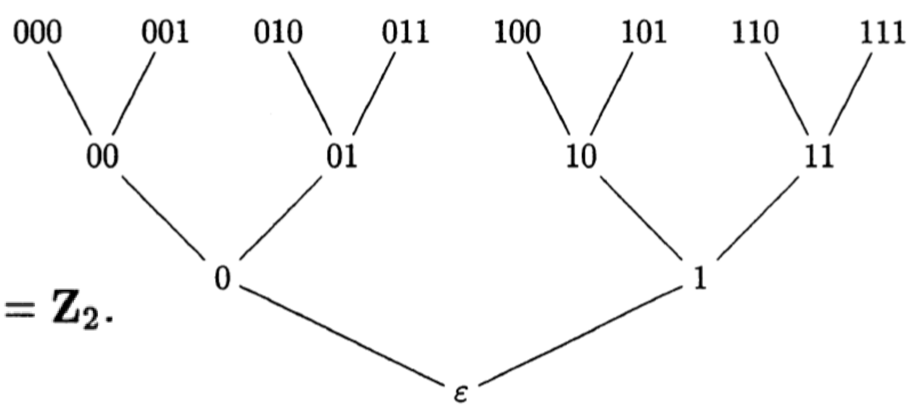



$T = \mathbf{Z}_2.$

# Constructing Instantaneous Codes (Cont.)

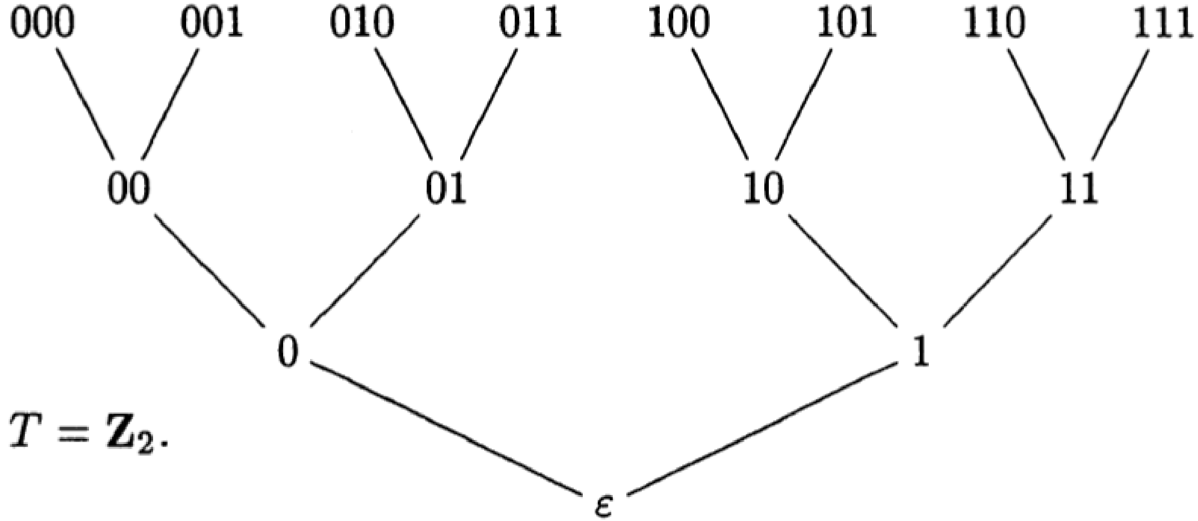
- A code $C$ can be regarded as a finite set of vertices of the tree $T^*$.
- A word $w_i$ is a prefix of $w_j$ if and only if the vertex $w_i$ is dominated by the vertex $w_j$
  - that is, there is an upward path in $T^*$ from $w_i$ to $w_j$
- $C$ is instantaneous if and only if no vertex $w_i \in C$ is dominated by a vertex $w_j \in C$ $(i \neq j)$.




$T = \mathbf{Z}_2.$

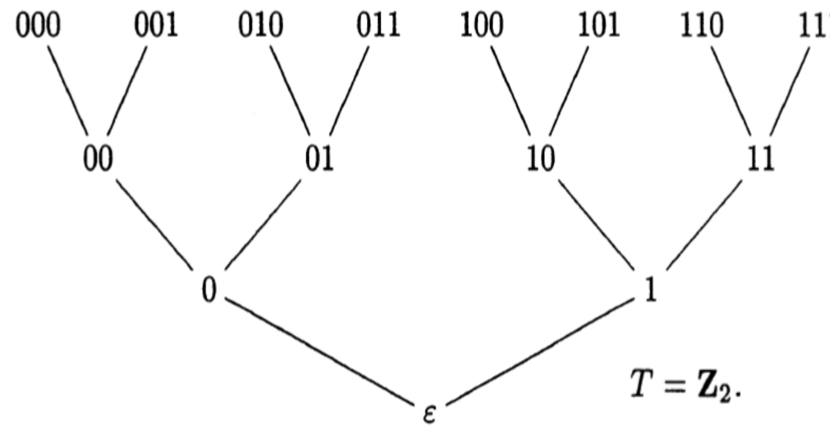
# Example 1.18

- Let us find an instantaneous **binary** code $C$ for a source $S$ with five symbols $s_1, \dots, s_5$.



# Example 1.19

- Is there an instantaneous **binary** code for this source $S$ with word-lengths 1, 2, 3, 3, 4?
- No, Why?




$T = \mathbf{Z}_2$.

- Is there an instantaneous **ternary** code for this source $S$ with word-lengths 1, 2, 3, 3, 4?
- Yes. Why?

# Example 1.19

- Is there an instantaneous **binary** code for this source $S$ with word-lengths 1, 2, 3, 3, 4?
- No, Why?

- Is there an instantaneous **ternary** code for this source $S$ with word-lengths 1, 2, 3, 3, 4?
- Yes. Why?