Coding and Information Theory
Chapter 4
Information Channels

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Chapter 4: Information Channels

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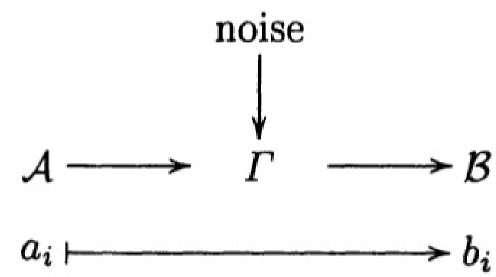
The aim of this chapter

- We Consider
  - a source sending messages through an unreliable (or noisy) channel to a receiver
- Our aim here is
  - to measure how much information is transmitted, and how much is lost in this process, using several different variations of the entropy function, and then
  - to relate this to the average word-length of the code used.

4.1 Notation and Definitions



- Information channel Γ
- Input of Γ: Source A,
  - with finite alphabet A of symbols $a = a_1, \dots, a_r$, having probabilities

$$p_i = \Pr(a = a_i)$$ where

$$0 \le p_i \le 1 \quad \text{and} \quad \sum_{i=1}^{r} p_i = 1$$

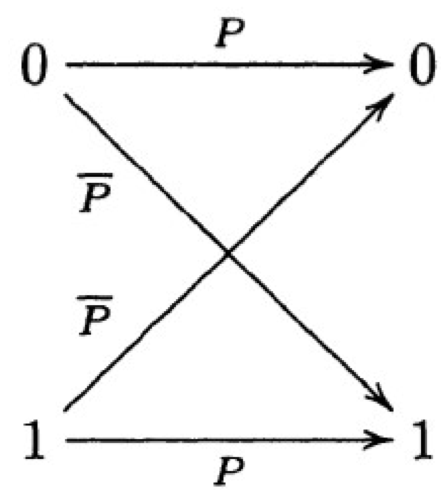
- Output of Γ: Source B,
  - with a finite alphabet B of symbols $b = b_1, \dots, b_s$, having probabilities

$$q_j = \Pr(b = b_j)$$ where

$$0 \le q_j \le 1 \quad \text{and} \quad \sum_{j=1}^{s} q_j = 1$$

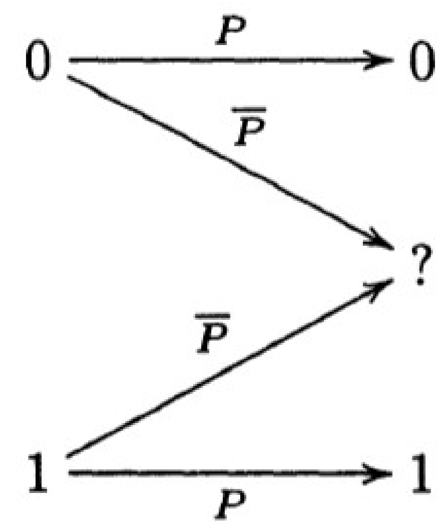
Example 4.1

- Binary symmetric channel (BSC)
  - $\mathrm{A} = \mathrm{B} = Z_2 = \{0, 1\}$.
  - Each input symbol $a = 0$ or 1 is correctly transmitted with probability P, and is incorrectly transmitted (as $\bar{a} = 1 - a$) with probability $\bar{P} = 1 - P$, for some constant P $(0 \leq P \leq 1)$.



Example 4.2

- Binary erasure channel (BEC)
  - $\mathrm{A} = Z_2 = \{0, 1\}$.
  - $\mathrm{B} = \{0, 1, ?\}$.
  - Each input symbol $a = 0$ or 1 is correctly transmitted with probability P, and is erased (or made illegible) with probability $\bar{P}$, indicated by an output symbol $b = ?$



Forward Probabilities

- Forward probabilities of Γ

$$P_{ij} = \Pr(b = b_j \mid a = a_i) = \Pr(b_j \mid a_i)$$

- We have $$\sum_{j=1}^{s} P_{ij} = 1$$

- The channel matrix $$M = (P_{ij}) = \begin{pmatrix} P_{11} & \dots & P_{1s} \\ \vdots & & \vdots \\ P_{r1} & \dots & P_{rs} \end{pmatrix}$$

- For instance, if Γ is the BSC or BEC we have $$M = \begin{pmatrix} P & \overline{P} \\ \overline{P} & P \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} P & 0 & \overline{P} \\ 0 & P & \overline{P} \end{pmatrix}$$

Combining two channels

- **Sum** $\Gamma + \Gamma'$
  - If Γ and Γ' have disjoint input alphabets A and A', and disjoint output alphabets B and B', then the **sum** $\Gamma + \Gamma'$ has input and output alphabets $A \cup A'$ and $B \cup B'$.
  - Each input symbol is transmitted through Γ or Γ', so the channel matrix is a block matrix

$$\begin{pmatrix} M & O \\ O & M' \end{pmatrix}$$

where M and M' are the channel matrices for Γ and Γ'.

Combining two channels

- **Product** $\Gamma \times \Gamma'$
  - The input and output alphabets are A x A' and B x B’
  - The sender transmits a pair $(a, a') \in$ A x A' by simultaneously sending a through Γ and a' through Γ'
  - A pair $(b, b') \in$ B X B' is received
  - Thus the forward probabilities are

$$\Pr((b, b') \mid (a, a')) = \Pr(b \mid a) \,.\, \Pr(b' \mid a')$$

  - So the channel matrix is the **Kronecker product** $M \otimes M'$ of the matrices M and M' for Γ and Γ'.
    - if $M = (P_{ij})$ and $M' = (P'_{kl})$ are $r \times s$ and $r' \times s'$ matrices, then $M \otimes M'$ is an $rr' \times ss'$ matrix, with entries $P_{ij}P'_{kl}$

Example

- If Γ and Γ' are binary symmetric channels, with channel matrices

$$M = \begin{pmatrix} P & \overline{P} \\ \overline{P} & P \end{pmatrix} \quad \text{and} \quad M' = \begin{pmatrix} P' & \overline{P'} \\ \overline{P'} & P' \end{pmatrix}$$

- then $\Gamma + \Gamma'$ and $\Gamma \times \Gamma'$ have channel matrices

$$\begin{array}{c} \\ 0 \\ 1 \\ 0' \\ 1' \end{array} \begin{array}{c} \begin{array}{cccc} 0, & 1, & 0', & 1' \end{array} \\ \begin{pmatrix} P & \overline{P} & 0 & 0 \\ \overline{P} & P & 0 & 0 \\ 0 & 0 & P' & \overline{P'} \\ 0 & 0 & \overline{P'} & P' \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{c} \begin{array}{cccc} (0,0'), & (1,0'), & (0,1'), & (1,1') \end{array} \\ \begin{pmatrix} PP' & \overline{P}P' & P\overline{P'} & \overline{P}\,\overline{P'} \\ \overline{P}P' & PP' & \overline{P}\,\overline{P'} & P\overline{P'} \\ P\overline{P'} & \overline{P}\,\overline{P'} & PP' & \overline{P}P' \\ \overline{P}\,\overline{P'} & P\overline{P'} & \overline{P}P' & PP' \end{pmatrix} \end{array} \begin{array}{c} \\ (0,0') \\ (1,0') \\ (0,1') \\ (1,1') \end{array}$$

The channel relationships

- The channel relationships

$$\sum_{i=1}^{r} p_i P_{ij} = q_j \qquad (4.2)$$

Where $p_i = \Pr(a = a_i)$, $q_j = \Pr(b = b_j)$ and

$$P_{ij} = \Pr\left(b = b_j \middle| a = a_i\right) = \Pr(b_j|a_i)$$

(4.2) can be written as $\mathbf{p}M = \mathbf{q}\,.\quad (4.2')$

- The backward probabilities

$$Q_{ij} = \Pr\left(a = a_i \mid b = b_j\right) = \Pr\left(a_i \mid b_j\right)$$

- The joint probabilities

$$R_{ij} = \Pr\left(a = a_i \text{ and } b = b_j\right) = \Pr\left(a_i, b_j\right)$$

Bayes' Formula

- **Bayes' Formula**

$$Q_{ij} = \frac{p_i}{q_j} P_{ij} \qquad (4.3)$$

provided $q_j \neq 0$.

- Combining this with (4.2) we get

$$Q_{ij} = \frac{p_i P_{ij}}{\sum_{k=1}^{r} p_k P_{kj}} \qquad (4.4)$$

4.2 The Binary Symmetric Channel

- Binary symmetric channel (BSC)
  - $\mathrm{A} = \mathrm{B} = Z_2 = \{0, 1\}$.
  - the channel matrix has the form

$$M = \begin{pmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{pmatrix} = \begin{pmatrix} P & \overline{P} \\ \overline{P} & P \end{pmatrix}$$

    for some P where $0 \leq P \leq 1$
  - The input probabilities have the form

$$p_0 = \Pr(a = 0) = p,$$

$$p_1 = \Pr(a = 1) = \overline{p},$$

    for some p such that $0 \leq p \leq 1$
  - The channel relationships = ? And Bayes' formula = ?

Examples

- Example 4.4
  - Let the input A be defined by putting $p = 1/2$
  - Probabilities of the output symbols: q_0 = ? And q_1 = ?
  - The backward probabilities: Q_{00}, Q_{01}, Q_{10}, Q_{11}, = ?
- Example 4.5
  - Suppose that P = 0.8 and p = 0.9
  - Probabilities of the output symbols: q_0 = ? And q_1 = ?
  - The backward probabilities: Q_{00}, Q_{01}, Q_{10}, Q_{11}, = ?
  - Necessary and sufficient conditions on p and P for

$$Q_{00} > Q_{10} \text{ and } Q_{01} > Q_{11}$$

4.3 System Entropies

- The input A and the output B of a channel Γ
  - the input entropy $H(\mathcal{A}) = \sum_i p_i \log \frac{1}{p_i}$
  - the output entropy $H(\mathcal{B}) = \sum_j q_j \log \frac{1}{q_j}$
  - If $b = b_j$ is received, there is a conditional entropy

$$H(\mathcal{A} \mid b_j) = \sum_i \Pr(a_i \mid b_j) \log \frac{1}{\Pr(a_i \mid b_j)} = \sum_i Q_{ij} \log \frac{1}{Q_{ij}}$$

  - the equivocation (of A with respect to B)

$$H(\mathcal{A} \mid \mathcal{B}) = \sum_j q_j H(\mathcal{A} \mid b_j) = \sum_j q_j \Big(\sum_i Q_{ij} \log \frac{1}{Q_{ij}} \Big) = \sum_i \sum_j R_{ij} \log \frac{1}{Q_{ij}}$$

System Entropies (Cont.)

- Similarly, if a_i is sent then the uncertainty about B is the conditional entropy

$$H(\mathcal{B} \mid a_i) = \sum_j \Pr(b_j \mid a_i) \log \frac{1}{\Pr(b_j \mid a_i)} = \sum_j P_{ij} \log \frac{1}{P_{ij}}$$

- the equivocation of B with respect to A

$$H(\mathcal{B} \mid \mathcal{A}) = \sum_i p_i H(\mathcal{B} \mid a_i) = \sum_i p_i \Big(\sum_j P_{ij} \log \frac{1}{P_{ij}} \Big) = \sum_i \sum_j R_{ij} \log \frac{1}{P_{ij}}$$

- the joint entropy

$$H(\mathcal{A}, \mathcal{B}) = \sum_i \sum_j \Pr(a_i, b_j) \log \frac{1}{\Pr(a_i, b_j)} = \sum_i \sum_j R_{ij} \log \frac{1}{R_{ij}}$$

System Entropies (Cont.)

- If A and B are statistically independent, then

$$H(\mathcal{A},\mathcal{B}) = H(\mathcal{A}) + H(\mathcal{B}) \qquad (4.5)$$

- In general, A and B are related, rather than independent, then

$$H(\mathcal{A},\mathcal{B}) = H(\mathcal{A}) + H(\mathcal{B} \mid \mathcal{A}) \qquad (4.6)$$

$$H(\mathcal{A},\mathcal{B}) = H(\mathcal{B}) + H(\mathcal{A} \mid \mathcal{B}) \qquad (4.7)$$

- We call $H(\mathrm{A}), H(\mathrm{B}), H(\mathrm{A}|\mathrm{B}), H(\mathrm{B}|\mathrm{A}), and\ H(\mathrm{A},\mathrm{B})$ the system entropies.

4.4 System Entropies for the Binary Symmetric Channel

- The input and output entropies for BSC are

$$H(\mathcal{A}) = -p \log p - \overline{p} \log \overline{p} = H(p),$$
$$H(\mathcal{B}) = -q \log q - \bar{q} \log \overline{q} = H(q),$$

  where $q = pP + \bar{p}\bar{P}$.

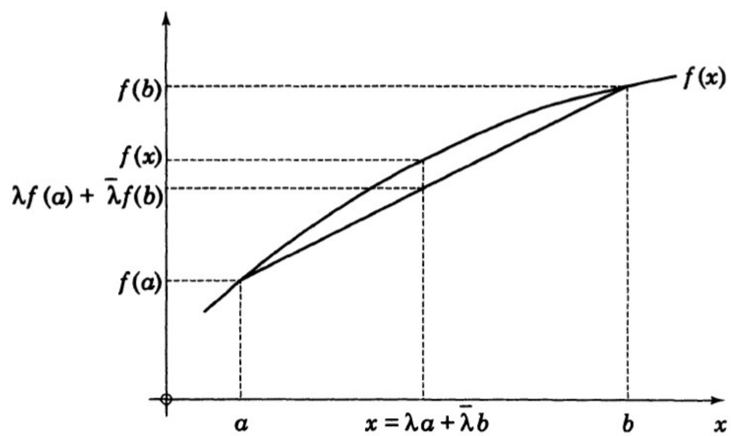
- Definition: A function f: [0,1] $\rightarrow R$ is strictly convex, if for $a, b \in [0,1]$ $and\ x = \lambda a + \bar{\lambda} b\ with\ 0 \leq \lambda \leq 1$,

$$f(x) \geq \lambda f(a) + \overline{\lambda} f(b)$$

  with equality if and only if x = a or b, that is, $a = b$ or $\lambda = 0\ or\ 1$.

System Entropies for BSC (Cont.)

- Lemma 4.6
  - If a function f : [0,1] $\rightarrow R$ is continuous on the interval [0,1] and twice differentiable on (0,1) , with f"(x) < 0 for all $x \in$ (0,1), then f is strictly convex.
- Corollary 4.7
  - The entropy function $H(p)$ is strictly convex on [0,1].



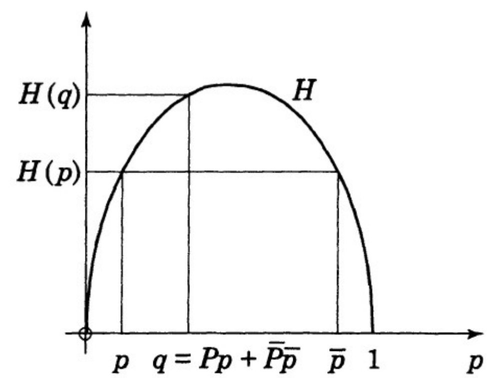


Figure 4.5

System Entropies for BSC (Cont.)

- The BSC satisfies

  $$H(\mathcal{B}) \geq H(\mathcal{A}), \qquad (4.8)$$

  with equality if and only if $p = 1/2$ or the channel is totally unreliable (P = 0) or reliable (P = 1)

Transmission through the BSC generally increases uncertainty

Note in BSC, $q = pP + \bar{p}\bar{P}$

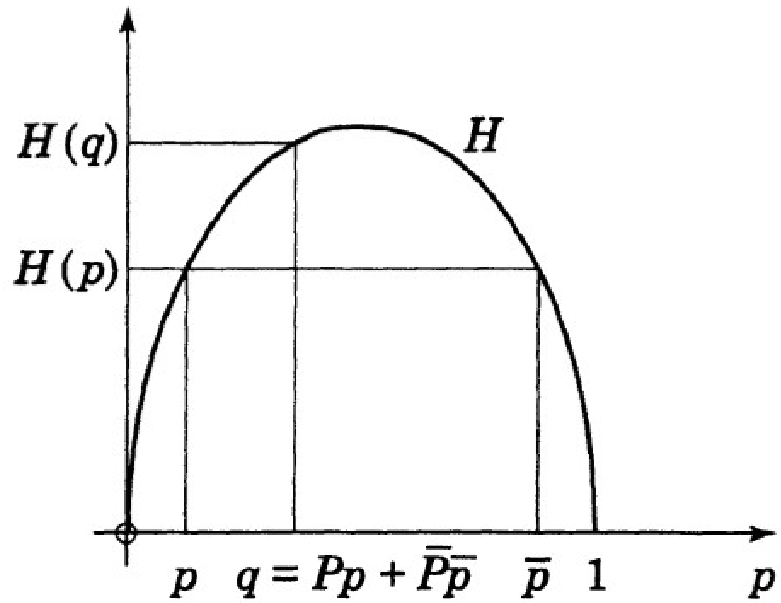


Figure 4.5

System Entropies for BSC (Cont.)

- For the BSC we have

$$H(\mathcal{B} \mid \mathcal{A}) = H(P)$$

- The equivocation for the BSC is

$$H(\mathcal{A} \mid \mathcal{B}) = H(p) + H(P) - H(q).$$

- The BSC satisfies

$$H(\mathcal{B} \mid \mathcal{A}) \leq H(\mathcal{B}), \qquad (4.9)$$

$$H(\mathcal{A} \mid \mathcal{B}) \leq H(\mathcal{A}), \qquad (4.10)$$

the uncertainty about B generally decreases when A is known

the uncertainty about A generally decreases when B is known

with equality if and only if $P = 1/2$ or $p = 0, 1$.

4.5 Extension of Shannon's First Theorem to Information Channels

- Extension of Shannon's First Theorem
  - The greatest lower bound of the average word-lengths of uniquely decodable encodings of the input A of a channel, given knowledge of its output B, is equal to the equivocation $H(\mathrm{A}|\mathrm{B})$.
- Interpretation
  - the receiver knows B but is uncertain about A; the extra information needed to be certain about A is the equivocation $H(\mathrm{A}|\mathrm{B})$, and
  - this is equal to the least average word-length required to supply that extra information (by some other means, separate from Γ).

Extension of Shannon's First Theorem

- Theorem 4.8
  - If the output B of a channel is known, then by encoding A^n with n sufficiently large, one can find uniquely decodable encodings of the input A with average word-lengths arbitrarily close to the equivocation $H(\mathrm{A}|\mathrm{B})$.

4.6 Mutual Information

- If Γ is a channel with input A and output B, then the entropy H(A) of A has three equivalent interpretations:
  1. it is the uncertainty about A when B is unknown;
  2. it is the information conveyed by A when B is unknown;
  3. it is the average word-length needed to encode A when B is unknown.
- Similarly, the equivocation H($\mathrm{A}|\mathrm{B}$) has three equivalent interpretations:
  1. it is the uncertainty about A when B is known;
  2. it is the information conveyed by A when B is known;
  3. it is the average word-length needed to encode A when B is known.

Mutual Information (Cont.)

- The mutual information is defined as the difference between these two numbers:

$$I(\mathcal{A}, \mathcal{B}) = H(\mathcal{A}) - H(\mathcal{A} \mid \mathcal{B})$$

- This also has three equivalent interpretations:
  1. it is the amount of uncertainty about A resolved by knowing B;
  2. it is the amount of information about A conveyed by B;
  3. it is the average number of symbols, in the code-words for A, which refer to B.

$I(\mathrm{A}, \mathrm{B})$ represents how much information A and B have in common

Examples

- Example 4.9
  - For a rather frivolous example, let Γ be a film company, A a book, and B the resulting film of the book. Then $I(\mathrm{A}, \mathrm{B})$ represents how much the film tells you about the book.
- Example 4.10
  - Let A be a lecture, Γ a student taking notes, and B the resulting set of lecture notes. Then $I(\mathrm{A}, \mathrm{B})$ measures how accurately the notes record the lecture.
- Interchanging the roles of A and B, we can define

$$I(\mathcal{B}, \mathcal{A}) = H(\mathcal{B}) - H(\mathcal{B} \mid \mathcal{A})$$

Mutual Information (Cont.)

- We have

$$I(\mathcal{A}, \mathcal{B}) = I(\mathcal{B}, \mathcal{A}) \quad (4.15)$$

$$I(\mathcal{A}, \mathcal{B}) = H(\mathcal{A}) + H(\mathcal{B}) - H(\mathcal{A}, \mathcal{B}) \quad (4.16)$$

- Theorem 4.11
  - For every channel Γ we have $I(\mathrm{A}, \mathrm{B}) \geq 0$, with equality if and only if the input A and the output B are statistically independent.

Mutual Information (Cont.)

- Corollary 4.12
  - For every channel Γ we have

$$H(\mathcal{A}) \geq H(\mathcal{A} \mid \mathcal{B}),$$
$$H(\mathcal{B}) \geq H(\mathcal{B} \mid \mathcal{A})$$
$$H(\mathcal{A}, \mathcal{B}) \leq H(\mathcal{A}) + H(\mathcal{B})$$

  - in each case, there is equality if and only if the input A and the output B are statistically independent.

4.7 Mutual Information for the Binary Symmetric Channel

- Let us take the channel Γ to be the BSC, we have

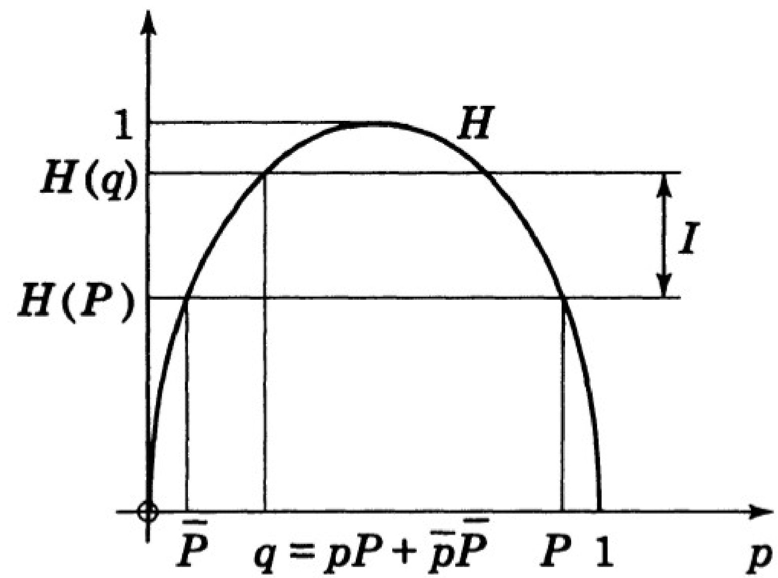
$$I(\mathcal{A},\mathcal{B}) = H(\mathcal{B}) - H(\mathcal{B} \mid \mathcal{A})$$

$$H(\mathcal{B}) = H(q) \text{ and } H(\mathcal{B} \mid \mathcal{A}) = H(P) \quad \text{where } q = pP + \overline{p}\overline{P}$$

- So that

$$\begin{aligned} &I(\mathcal{A},\mathcal{B}) \\ &= H(q) - H(P) \\ &= H(pP + \overline{p}\overline{P}) - H(P) \end{aligned}$$

$$\boxed{0 \le I(\mathcal{A},\mathcal{B}) \le 1 - H(P)}$$



4.8 Channel Capacity

- The mutual information $I(\mathrm{A}, \mathrm{B})$ for a channel Γ represents how much of the information in the input A is emerging in the output B.
    - This depends on both Γ and A
- The capacity C of a channel Γ is defined to be the maximum value of the mutual information $I(\mathrm{A}, \mathrm{B})$, where A ranges over all possible inputs for Γ.
    - This depends on Γ alone, represents the maximum amount of information which the channel can transmit

Channel Capacity (Cont.)

- Example 4.13
  - We saw that the BSC has channel capacity C = 1 - H(P), attained when the input satisfies p = 1/2.
  - Figure shows C as a function of P
    - C is greatest when P is 0 or 1
    - C is least when P = 1/2