CS 4450

Coding and Information Theory

Overview of Probability (A)

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Overview of Probability

1. Fundamentals of Probability
2. Random Variables and its Characteristics
3. Statistical Averages

1. Fundamentals of Probability

- Probability: a measure of certainty
- Experiment: any process of observation.
- Outcome: the result of an experiment.
- Random Experiment:
  - An experiment whose outcome is not unique and therefore cannot be predicted with certainty
- Sample Space, Sample Point, and Event
  - The set S of all possible outcomes of a given random experiment is called the sample space, or universal set.
  - An element (outcome) in S is known as a sample point or elementary event.
  - An event A is a subset of a sample space ($A \subset S$)

Defining Probability

The probability of an event can be defined by the following two approaches:

1. Mathematical or Classical Probability

   If an event can happen in m different ways out of a total number of n possible ways, all of which are equally likely, then the probability of that event is m/n.

2. Statistical or Empirical or Estimated Probability

   If an experiment is repeated n times (n being very large) under homogeneous and identical conditions and an event is observed to occur m number of times, then the probability of the event is $\lim_{n \to \infty} {}^{m}/_{n}$.

1.1 Operations of Events

We can combine events to form new events using various set operations as follows

1. The complement of event A: ($\bar{A}$).
2. The union of events A and B: ($A \cup B$).
3. The intersection of events A and B: ($A \cap B$).

Note that

1. Null or impossible event is represented by empty set $\emptyset$.
2. Two events A and B are called mutually exclusive or disjoint if $A \cap B = \emptyset$

Example 1.1: Throw a die and observe the number of dots appearing on the top face.

(a) Construct its sample space.

(b) If A be the event that an odd number occurs, B that an even number occurs, and C that a prime number occurs, write down their respective subsets.

(c) Find the event that an even or a prime number occurs.

(d) List the outcomes for the event that an even prime number occurs.

(e) Find the event that a prime number does not occur.

(f) Find the event that seven dots appear on the top face.

(g) Find the event that even and odd numbers occur simultaneously

1.2 Axioms of Probability

Let P(A) denote the probability of event A of some sample space S. It must satisfy the following axioms:

- $0 \leq P(A) \leq 1$ (1.1)
- $P(S) = 1$ (1.2)
- $P(A \cup B) = P(A) + P(B)$, if $A \cap B = \emptyset$ (1.3a)
- $P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$, if all $A_i \cap A_j = \emptyset$ (1.3b)

1.3 Properties of Probability

Some important properties on probability are as follows:

- $P(\emptyset) = 0$ (1.4)
- $P(\overline{A}) = 1 - P(A)$ (1.5)
- $P(A) \leq P(B)$, if $A \subset B$ (1.6)
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (1.7a)
- $P(A \cup B) \leq P(A) + P(B)$, (1.7b)
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$ (1.8)

Examples

Example 1.2: Determine the probability for the event that the sum 8 appears in a single throw of a pair of fair dice.

Example 1.3: A ball is drawn at random from a box containing 5 red balls, 4 green balls, and 3 blue balls. Determine the probability that it is (a) green, (b) not green, and (c) red or blue.

Example 1.4: Two cards are drawn at random from a pack of 52 cards. Find the probability that (a) both are hearts and (b) one is heart and one is spade.

Example 1.5: A telegraph source emits two symbols, dash and dot. It was observed that the dash were thrice as likely to occur as dots. Find the probabilities of the dashes and dots occurring.

Example 1.6: A digital message is 1000 symbols long. An average of 0.1% symbols received is erroneous. (a) What is the probability of correct reception? (b) How many symbols are correctly received?

1.4 Conditional Probability

The conditional probability of an event A given that B has happened (P(A/B)) is defined as follows:

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \text{ if } P(B) > 0 \qquad (1.9)$$

Similarly,

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \text{ if } P(A) > 0 \qquad (1.10)$$

Using Equations (1.9) and (1.10) we can write

$$P(A \cap B) = P(A/B)P(B) = P(B/A)P(A) \qquad (1.11)$$

or

$$P(A/B) = \frac{P(B/A)P(A)}{P(B)} \qquad (1.12)$$

This is known as Bayes rule.

Example

Example 1.7: Consider the following table:

Sex	**Employed (E)**	**Unemployed (U)**	**Total**
Male (M)	250	50	300
Female (F)	150	100	250
Total	400	150	550

(a) If a person is male, what is the probability that he is unemployed?
(b) If a person is female, what is the probability that she is employed?
(c) If a person is employed, what is the probability that he is male?

1.5 Independent Events

Two events A and B are said to be independent if

$$P(B/A) = P(B) \tag{1.13}$$

or

$$P(A/B) = P(A) \tag{1.14}$$

Combining Equations (1.11) and (1.13), we have

$$P(A \cap B) = P(A)P(B) \tag{1.15}$$

Example 1.8: Determine the probability for the event that at least one head appears in three tosses of a fair coin

1.6 Total Probability

Let A_1, A_2, …, A_n be mutually exclusive ($A_i \cap A_j = \emptyset$, for $i \neq j$) and exhaustive ($\bigcup_{i=1}^{n} A_n = S$). Now B is any event in S. Then

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B/A_i)P(A_i) \qquad (1.16)$$

This is known as the total probability of event B.

If $A = A_i$ in Equation (1.12), then using (1.16) we have the following important theorem (Bayes theorem):

$$P(A_i/B) = \frac{P(B/A_i)P(A_i)}{\sum_{i=1}^{n} P(B/A_i)P(A_i)} \qquad (1.17)$$

Recall $P(A/B) = \frac{P(B/A)P(A)}{P(B)} \qquad (1.12)$