Coding and Information Theory Chapter 4 Information Channels - A

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Quick Review of Last Lecture

- Shannon-Fane Coding examples

$$l_i = \lceil \log_2(1/p_i) \rceil = \min\{n \in \mathbf{Z} \mid 2^n \geq 1/p_i\}$$

- Entropy of Extensions and Products

$$H_r(S^n) = nH_r(S).$$

- Shannon's First Theorem

$$\lim_{n\to\infty} \frac{L_n}{n} = H_r(S)\,.$$

- An Example of Shannon's First Theorem

S has two symbols s_1, s_2 of probabilities $p_i = 2/3,\ 1/3$

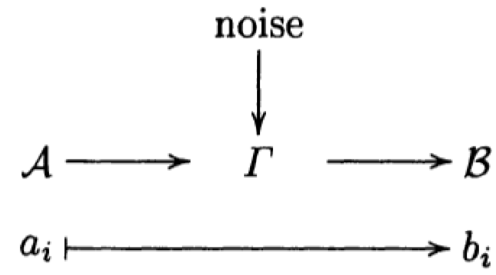
Chapter 4: Information Channels

1. Notation and Definitions
2. The Binary Symmetric Channel
3. System Entropies
4. System Entropies for the Binary Symmetric Channel
5. Extension of Shannon's First Theorem to Information Channels
6. Mutual Information
7. Mutual Information for the Binary Symmetric Channel
8. Channel Capacity

The aim of this chapter

- We Consider
  - a source sending messages through an unreliable (or noisy) channel to a receiver
- Our aim here is
  - to measure how much information is transmitted, and how much is lost in this process, using several different variations of the entropy function, and then
  - to relate this to the average word-length of the code used.

4.1 Notation and Definitions



- Information channel Γ
- Input of Γ: Source A,
  - with finite alphabet A of symbols $a = a_1, \ldots, a_r$, having probabilities

$$p_i = \Pr(a = a_i) \quad \text{where}$$

$$0 \leq p_i \leq 1 \qquad \text{and} \qquad \sum_{i=1}^{r} p_i = 1$$

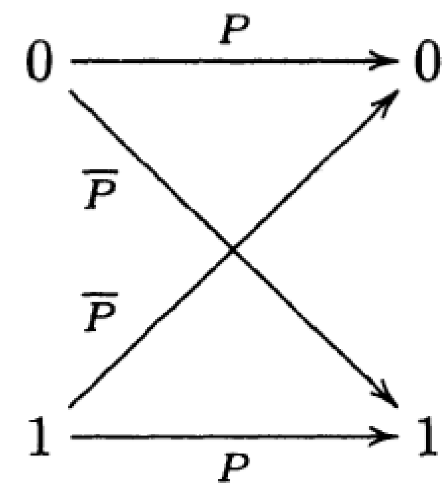
- Output of Γ: Source B,
  - with a finite alphabet B of symbols $b = b_1, \ldots, b_s$, having probabilities

$$q_j = \Pr(b = b_j) \quad \text{where}$$

$$0 \leq q_j \leq 1 \qquad \text{and} \qquad \sum_{j=1}^{s} q_j = 1$$

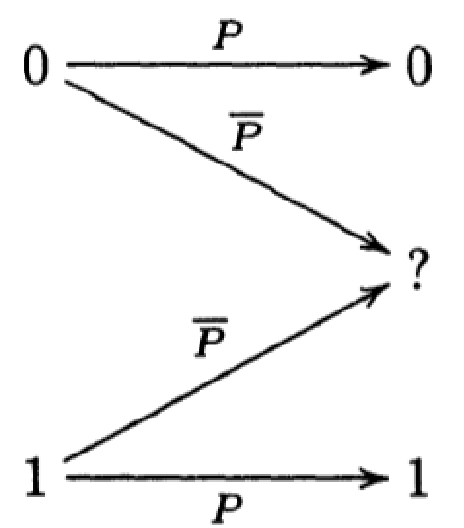
Example 4.1

- Binary symmetric channel (BSC)
  - $\mathrm{A} = \mathrm{B} = Z_2 = \{0, 1\}$.
  - Each input symbol $a = 0$ or 1 is correctly transmitted with probability P, and is incorrectly transmitted (as $\bar{a} = 1 - a$) with probability $\bar{P} = 1 - P$, for some constant P $(0 \leq P \leq 1)$.



Example 4.2

- Binary erasure channel (BEC)
  - $\mathrm{A} = Z_2 = \{0, 1\}$.
  - $\mathrm{B} = \{0, 1, ?\}$.
  - Each input symbol $a = 0$ or 1 is correctly transmitted with probability P, and is erased (or made illegible) with probability $\bar{P}$, indicated by an output symbol $b = ?$



Forward Probabilities

- Forward probabilities of Γ

$$P_{ij} = \Pr(b = b_j \mid a = a_i) = \Pr(b_j \mid a_i)$$

- We have $\sum_{j=1}^{s} P_{ij} = 1$

- The channel matrix $M = (P_{ij}) = \begin{pmatrix} P_{11} & \dots & P_{1s} \\ \vdots & & \vdots \\ P_{r1} & \dots & P_{rs} \end{pmatrix}$

Forward Probabilities – Binary channel

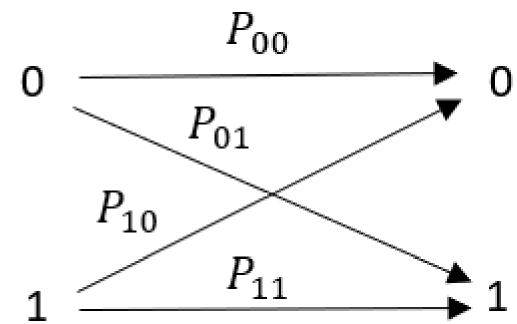
Channel:
r input symbols
s output symbols

$$M = (P_{ij}) = \begin{pmatrix} P_{11} & \dots & P_{1s} \\ \vdots & & \vdots \\ P_{r1} & \dots & P_{rs} \end{pmatrix}$$

$$\sum_{j=1}^{s} P_{ij} = 1$$

$$P_{ij} = \Pr\left(b = b_j \mid a = a_i\right) = \Pr\left(b_j \mid a_i\right)$$

Binary channel



$$M = \begin{pmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{pmatrix}$$

$$P_{00} = \Pr(b = 0 \mid a = 0)$$

$$P_{01} = \Pr(b = 1 \mid a = 0)$$

$$P_{10} = \Pr(b = 0 \mid a = 1)$$

$$P_{11} = \Pr(b = 1 \mid a = 1)$$

Forward Probabilities – BSC

Channel:
r input symbols
s output symbols

$$M = (P_{ij}) = \begin{pmatrix} P_{11} & \dots & P_{1s} \\ \vdots & & \vdots \\ P_{r1} & \dots & P_{rs} \end{pmatrix}$$

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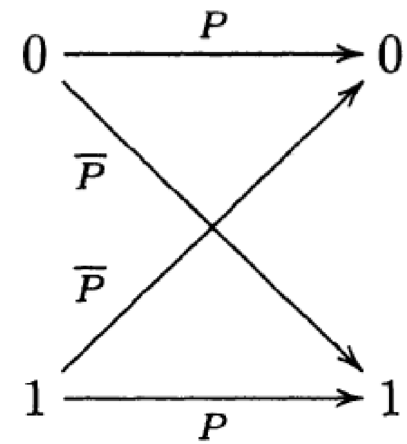
BSC



$$M = \begin{pmatrix} P & \overline{P} \\ \overline{P} & P \end{pmatrix}$$

Forward Probabilities – BEC

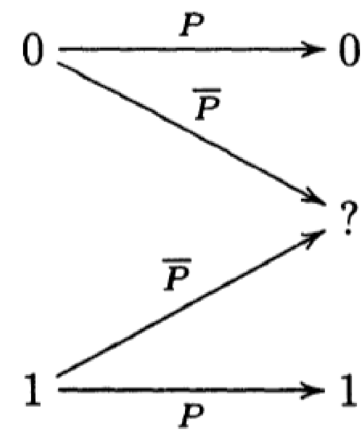
Channel:
r input symbols
s output symbols

$$M = (P_{ij}) = \begin{pmatrix} P_{11} & \dots & P_{1s} \\ \vdots & & \vdots \\ P_{r1} & \dots & P_{rs} \end{pmatrix}$$

$$\sum_{j=1}^{s} P_{ij} = 1$$

$$P_{ij} = \Pr(b = b_j \mid a = a_i) = \Pr(b_j \mid a_i)$$

BEC



$$M = \begin{pmatrix} P & 0 & \overline{P} \\ 0 & P & \overline{P} \end{pmatrix}$$

Combining two channels

- **Sum** $\Gamma + \Gamma'$
  - If Γ and Γ' have disjoint input alphabets A and A', and disjoint output alphabets B and B', then the **sum** $\Gamma + \Gamma'$ has input and output alphabets $A \cup A'$ and $B \cup B'$.
  - Each input symbol is transmitted through Γ or Γ', so the channel matrix is a block matrix

$$\begin{pmatrix} M & O \\ O & M' \end{pmatrix}$$

where M and M' are the channel matrices for Γ and Γ'.

The channel relationships

- The channel relationships

$$\sum_{i=1}^{r} p_i P_{ij} = q_j \qquad (4.2)$$

Where $p_i = \Pr(a = a_i)$, $q_j = \Pr(b = b_j)$ and

$$P_{ij} = \Pr\left(b = b_j \middle| a = a_i\right) = \Pr(b_j|a_i)$$

The channel relationships: Cont.

- The channel relationships

$$\sum_{i=1}^{r} p_i P_{ij} = q_j \qquad (4.2)$$

(4.2) can be written as

$$\mathbf{p}M = \mathbf{q}\,. \qquad (4.2')$$

- The backward probabilities

$$Q_{ij} = \Pr(a = a_i \mid b = b_j) = \Pr(a_i \mid b_j)$$

- The joint probabilities

$$R_{ij} = \Pr(a = a_i \text{ and } b = b_j) = \Pr(a_i, b_j)$$

Bayes' Formula

- **Bayes' Formula**

$$Q_{ij} = \frac{p_i}{q_j} P_{ij} \qquad (4.3)$$

provided $q_j \neq 0$.

$$\begin{aligned} p_i P_{ij} &= \Pr(a_i)\Pr(b_j \mid a_i) \\ &= \Pr(a_i, b_j) = R_{ij} \end{aligned}$$

$$\begin{aligned} q_j Q_{ij} &= \Pr(b_j)\Pr(a_i \mid b_j) \\ &= \Pr(a_i, b_j) = R_{ij} \end{aligned}$$

- Combining this with (4.2) we get

$$Q_{ij} = \frac{p_i P_{ij}}{\sum_{k=1}^{r} p_k P_{kj}} \qquad (4.4)$$

$$\sum_{i=1}^{r} p_i P_{ij} = q_j \qquad (4.2)$$

Example: In a binary communication system below. Given

$p_0 = P(a = 0) = 0.5,$

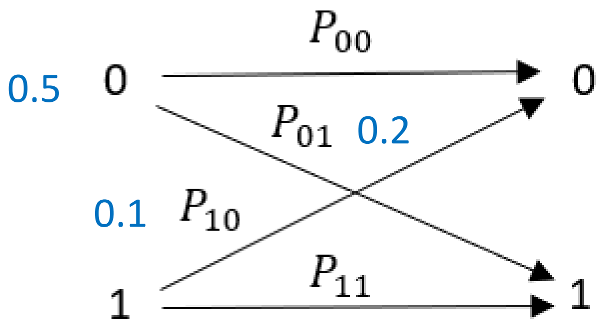
$P_{01} = P(b = 1 \mid a = 0) = 0.2$ and

$P_{10} = P(a = 0 \mid b = 1) = 0.1,$

(a) Find $q_0 = P(b = 0)$ and $q_1 = P(b = 1)$.

(b) Find $Q_{11} = P(a = 1 \mid b = 1)$

(c) Find $Q_{00} = P(a = 0 \mid b = 0)$



To solve (a) Using the channel relationships formular (4.2)

$$q_0 = P_{00} \times p_0 + P_{10} \times p_1 = 0.8 \times 0.5 + 0.1 \times 0.5 = 0.45$$

$$q_1 = P_{01} \times p_0 + P_{11} \times p_1 = 0.2 \times 0.5 + 0.9 \times 0.5 = 0.55$$

Example: In a binary communication system below. Given

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(a) Find $q_0 = P(b = 0)$ and $q_1 = P(b = 1)$.

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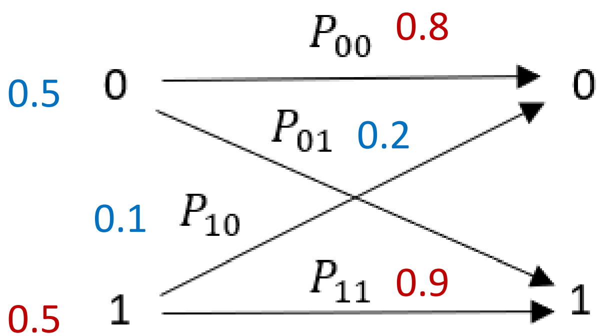

To solve (b) and (c) Using Bayes rule (4.3)

$$Q_{11} = \frac{p_1 P_{11}}{q_1} = \frac{0.5 \times 0.9}{0.55} = 0.818$$

$$Q_{00} = \frac{p_0 P_{00}}{q_0} = \frac{0.5 \times 0.8}{0.45} = 0.889$$