Chapter 7: Space-for-time tradeoffs

Two varieties of space-for-time algorithms:

- ***input enhancement*** — **preprocess the input (or its part) to store some info to be used later in solving the problem**
  - **counting methods for sorting**
  - **string searching algorithms**

- ***prestructuring*** — **preprocess the input to make accessing its elements easier**
  - **hashing**
  - **indexing schemes (e.g., B-trees)**

7.1 Sorting by Counting

- **Comparison-counting Sort**
  - **for each element of a list to be sorted, count the total number of elements smaller than this element and record the results in a table**
- **Example of sorting by comparison counting**

Array *A*[0..5]		62	31	84	96	19	47
Initially	*Count* []	0	0	0	0	0	0
After pass *i* = 0	*Count* []	3	0	1	1	0	0
After pass *i* = 1	*Count* []		1	2	2	0	1
After pass *i* = 2	*Count* []			4	3	0	1
After pass *i* = 3	*Count* []				5	0	1
After pass *i* = 4	*Count* []					0	2
Final state	*Count* []	3	1	4	5	0	2
Array *S*[0..5]		19	31	47	62	84	96

Seudocode of Comparison-counting Sort

```
ALGORITHM  ComparisonCountingSort(A[0..n − 1])
    //Sorts an array by comparison counting
    //Input: An array A[0..n − 1] of orderable elements
    //Output: Array S[0..n − 1] of A's elements sorted in nondecreasing order
    for i ← 0 to n − 1 do Count[i] ← 0
    for i ← 0 to n − 2 do
        for j ← i + 1 to n − 1 do
            if A[i] < A[j]
                Count[j] ← Count[j] + 1
            else Count[i] ← Count[i] + 1
    for i ← 0 to n − 1 do S[Count[i]] ← A[i]
    return S
```

- **time efficiency $\Theta(n^2)$: is the same as the selection sort**

Sorting by distribution counting

- **EXAMPLE:**
  - **Consider sorting the array: 13, 11, 12, 13, 12, 12**
  - **Compute frequencies and distribution:**

    Distribution value indicates position of last occurrence of the array value in the sorted array.

Array values	11	12	13
Frequencies	1	3	2
Distribution values	1	4	6

  - **Process the array from right to left**

    put each array value in the position indicated by distribution value and reduce the distribution value by 1

	D[0..2]			S[0..5]					
A [5] = 12	1	**4**	6				12		
A [4] = 12	1	**3**	6			12			
A [3] = 13	1	2	**6**						13
A [2] = 12	1	**2**	5		12				
A [1] = 11	**1**	1	5	11					
A [0] = 13	0	1	**5**					13	

Pseudocode of distribution counting

ALGORITHM *DistributionCountingSort*($A[0..n-1], l, u$)

```
//Sorts an array of integers from a limited range by distribution counting
//Input: An array A[0..n − 1] of integers between l and u (l ≤ u)
//Output: Array S[0..n − 1] of A's elements sorted in nondecreasing order
for j ← 0 to u − l do D[j] ← 0                          //initialize frequencies
for i ← 0 to n − 1 do D[A[i] − l] ← D[A[i] − l] + 1     //compute frequencies
for j ← 1 to u − l do D[j] ← D[j − 1] + D[j]            //reuse for distribution
for i ← n − 1 downto 0 do
    j ← A[i] − l
    S[D[j] − 1] ← A[i]
    D[j] ← D[j] − 1
return S
```

- **Time efficiency: $\Theta(n)$**

7.2 Review: String searching by brute force

pattern**: a string of *****m***** characters to search for**

text**: a (long) string of *****n***** characters to search in**

<u>Brute force algorithm</u>

Step 1 Align pattern at beginning of text

Step 2 Moving from left to right, compare each character of pattern to the corresponding character in text until either all characters are found to match (successful search) or a mismatch is detected

Step 3 While a mismatch is detected and the text is not yet exhausted, realign pattern one position to the right and repeat Step 2

String searching by preprocessing

Several string searching algorithms are based on the input enhancement idea of preprocessing the pattern

- **Knuth-Morris-Pratt (KMP) algorithm preprocesses pattern left to right to get useful information for later searching**
- **Boyer -Moore algorithm preprocesses pattern right to left and store information into two tables**
- **Horspool's algorithm simplifies the Boyer-Moore algorithm by using just one table**

Horspool's Algorithm

A simplified version of Boyer-Moore algorithm:

- **preprocesses pattern to generate a shift table that determines how much to shift the pattern when a mismatch occurs**

- **always makes a shift based on the text's character *c* aligned with the last character in the pattern according to the shift table's entry for *c***

$s_0 \quad \dots \quad c \quad \dots \quad s_{n-1}$

B A R B E R

How far to shift?

Look at first (rightmost) character in text that was compared:

- **The character is not in the pattern**

  `.....c......................` (`c` not in pattern)

  `BAOBAB`

- **The character is in the pattern (but not the rightmost)**

  `.....O......................` (`O` occurs once in pattern)

  `BAOBAB`

  `.....A......................` (`A` occurs twice in pattern)

  `BAOBAB`

- **The rightmost characters do match**

  `.....B......................`

  `BAOBAB`

Shift table

- Shift sizes can be precomputed by the formula

$$t(c) = \begin{cases} \text{distance from } c\text{'s rightmost occurrence in pattern} \\ \quad \text{among its first } m\text{-1 characters to its right end} \\ \text{pattern's length } m, \text{ otherwise} \end{cases}$$

by scanning pattern before search begins and stored in a table called *shift table*

- Shift table is indexed by text and pattern alphabet
Eg, for BAOBAB :

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	2	6	6	6	6	6	6	6	6	6	6	6	6	3	6	6	6	6	6	6	6	6	6	6	6

Example of Horspool's alg. application

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	_
1	2	6	6	6	6	6	6	6	6	6	6	6	6	3	6	6	6	6	6	6	6	6	6	6	6	6

```
BARD LOVED BANANAS
BAOBAB
      BAOBAB
        BAOBAB
               BAOBAB  (unsuccessful search)
```

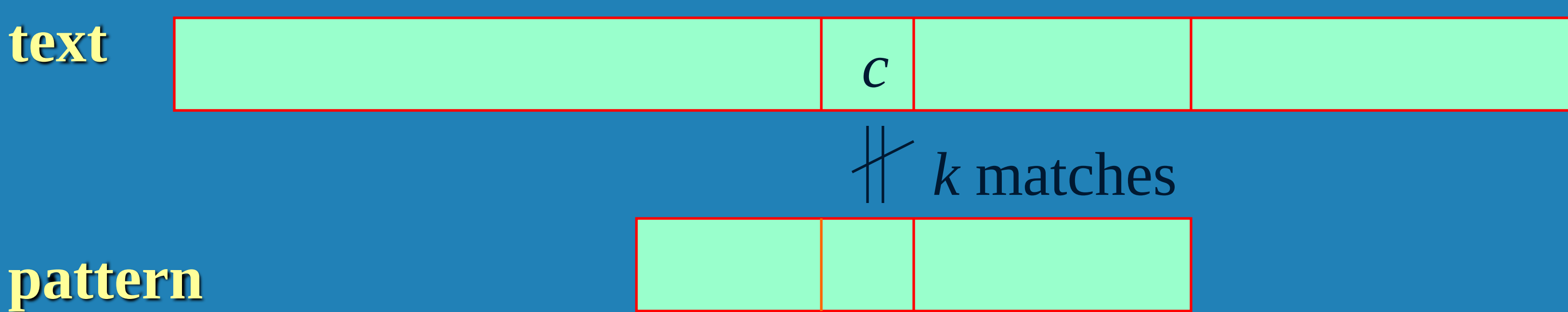

Boyer-Moore algorithm

Based on same two ideas:

- **comparing pattern characters to text from right to left**
- **precomputing shift sizes in two tables**
  - ***bad-symbol table*** **indicates how much to shift based on text's character causing a mismatch**
  - ***good-suffix table*** **indicates how much to shift based on matched part (suffix) of the pattern**

Bad-symbol shift in Boyer-Moore algorithm

- If the rightmost character of the pattern doesn't match, BM algorithm acts as Horspool's
- If the rightmost character of the pattern does match, BM compares preceding characters right to left until either all pattern's characters match or a mismatch on text's character *c* is encountered after $k > 0$ matches

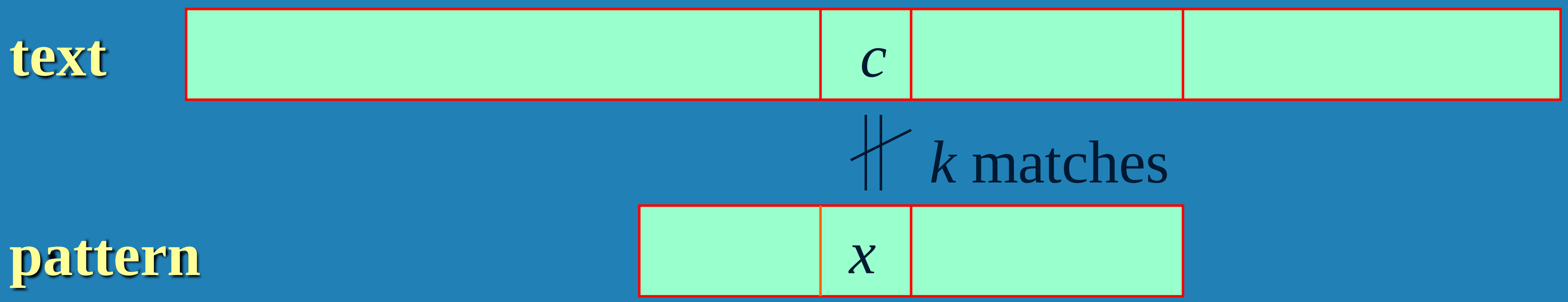


- bad-symbol shift $d_1 = \max\{t_1(c) - k, 1\}$, where $t_1(c)$ is pre-computed by Horspool's algorithm

Good-suffix shift in Boyer-Moore algorithm

- **Good-suffix shift d_2 is applied after $0 < k < m$ last characters were matched**
- **$d_2(k)$ = the distance between matched suffix of size k and its rightmost occurrence in the pattern that is not preceded by the same character as the matched suffix**

Example: `WOWWOW` $d_2(1) = 2$, $d_2(3) = 3$

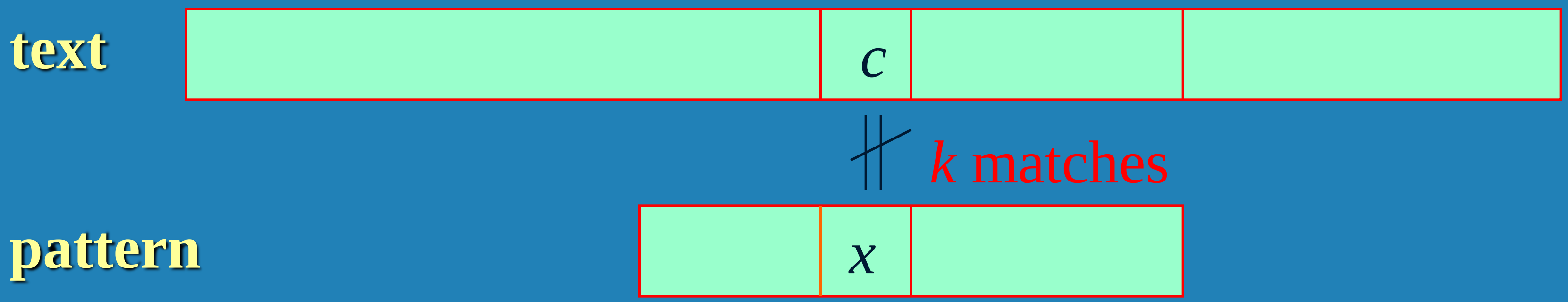


Good-suffix shift in Boyer-Moore algorithm

- If there is no such occurrence, match the longest suffix of **the matched k-character suffix of the pattern** with corresponding prefix of the pattern;

Example: `WOWWOW` $d_2(2) = 5$, $d_2(4) = 5$, $d_2(5) = 5$

- If there are no such suffix-prefix matches, $d_2(k) = m$



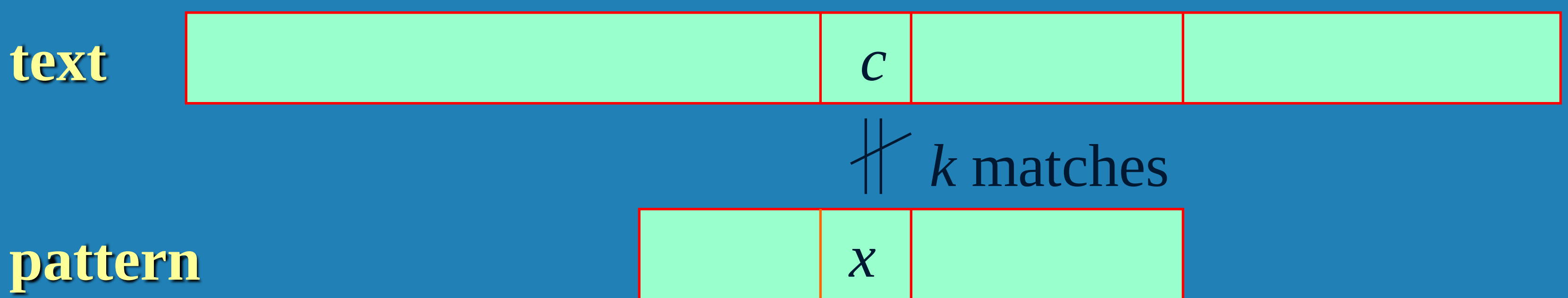
Good-suffix shift in the Boyer-Moore alg. (cont.)

After matching successfully $0 < k < m$ characters, the algorithm shifts the pattern right by

$$d = \max\{d_1, d_2\}$$

where $d_1 = \max\{t_1(c) - k, 1\}$ is bad-symbol shift

$d_2(k)$ is good-suffix shift



Boyer-Moore Algorithm (cont.)

Step 1 Fill in the bad-symbol shift table

Step 2 Fill in the good-suffix shift table

Step 3 Align the pattern against the beginning of the text

Step 4 Repeat until a matching substring is found or text ends:

Compare the corresponding characters right to left.

If no characters match, retrieve entry $t_1(c)$ from the bad-symbol table for the text's character c causing the mismatch and shift the pattern to the right by $t_1(c)$.
If $0 < k < m$ characters are matched, retrieve entry $t_1(c)$ from the bad-symbol table for the text's character c causing the mismatch and entry $d_2(k)$ from the good-suffix table and shift the pattern to the right by

$$d = \max \{d_1, d_2\}$$

where $d_1 = \max\{t_1(c) - k, 1\}$.

Example of Boyer-Moore alg. application

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	_
1	2	6	6	6	6	6	6	6	6	6	6	6	6	3	6	6	6	6	6	6	6	6	6	6	6	6

B E S S _ K N E W _ A B O U T _ B A O B A B S

B A O B A B

$d_1 = t_1(\text{K}) = 6$

B A O B A B

$d_1 = t_1(\_)-2 = 4$

$\underline{d_2(2) = 5}$

B A O B A B

$\underline{d_1 = t_1(\_)-1 = 5}$

$d_2(1) = 2$

B A O B A B (success)

k	pattern	d_2
1	BAO**B**A**B**	2
2	**B**AOB**AB**	5
3	**B**AO**BAB**	5
4	**B**A**OBAB**	5
5	**BAOBAB**	5

Boyer-Moore example from their paper

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	_
1	7	7	7	7	7	7	2	7	7	7	7	7	7	3	7	7	7	7	3	7	7	7	7	7	7	4

Find pattern `AT_THAT` in

`WHICH_FINALLY_HALTS. _ _AT_THAT`

k	pattern	d_2
1	AT_THAT	3
2	AT_THAT	5
3	AT_THAT	5
4	AT_THAT	5
5	AT_THAT	5
6	AT_THAT	5

Boyer-Moore example from exercise

How many character comparisons will the Boyer-Moore algorithm make in searching for the pattern 01010 in the binary text of 1000 zeros?

the bad-symbol table

c	0	1
$t_1(c)$	2	1

01010

the good-suffix table

k	the pattern	d_2
1	01010	4
2	01010	4
3	01010	2
4	01010	2

7.3 Hashing

- A very efficient method for implementing a *dictionary*, i.e., a set with the operations:
  - find
  - insert
  - delete

- Based on representation-change and space-for-time tradeoff ideas

- Important applications:
  - symbol tables
  - databases (*extendible hashing*)

Hash tables and hash functions

The idea of *hashing* is to map keys of a given file of size *n* into a table of size *m*, called the *hash table*, by using a predefined function, called the *hash function*,

$$h: K \rightarrow \text{ location (cell) in the hash table}$$

Example: student records, key = SSN. Hash function:
$h(K) = K \bmod m$ where m is some integer (typically, prime)
If $m = 1000$, where is record with SSN= 314159265 stored?

Generally, a hash function should:

- **be easy to compute**
- **distribute keys about evenly throughout the hash table**

Collisions

If $h(K_1) = h(K_2)$, there is a *collision*

- **Good hash functions result in fewer collisions but some collisions should be expected (*birthday paradox*)**

- **Two principal hashing schemes handle collisions differently:**
  - ***Open hashing***
    – each cell is a header of linked list of all keys hashed to it
  - ***Closed hashing***
    – one key per cell
    – in case of collision, finds another cell by
      – *linear probing:* use next free bucket
      – *double hashing:* use second hash function to compute increment

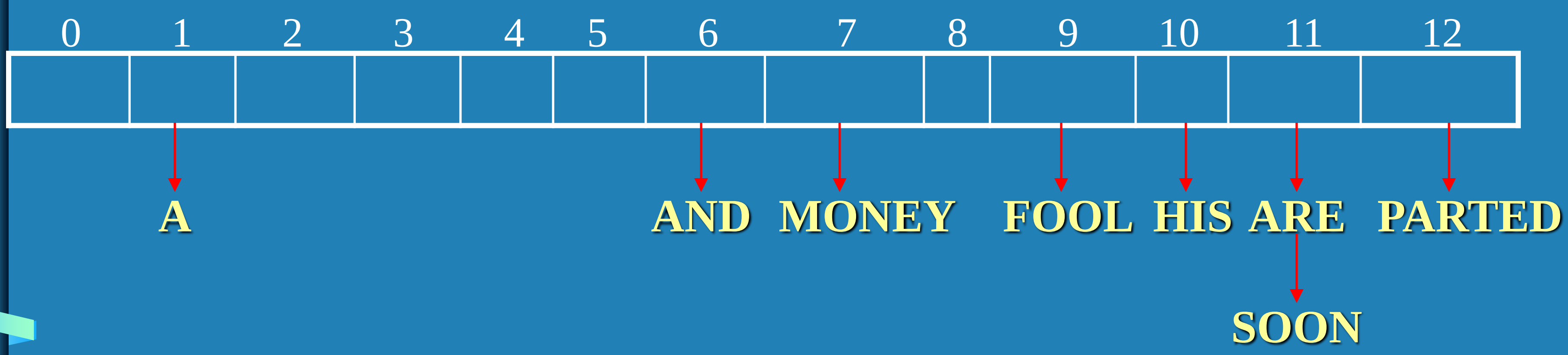
Open hashing (Separate chaining)

Keys are stored in linked lists <u>outside</u> a hash table whose elements serve as the lists' headers.

Example: A, FOOL, AND, HIS, MONEY, ARE, SOON, PARTED

***h*(*K*) = sum of *K* 's letters' positions in the alphabet MOD 13**

Key	A	FOOL	AND	HIS	MONEY	ARE	SOON	PARTED
h(*K*)	1	9	6	10	7	11	11	12



Search for KID

Open hashing (cont.)

- **If hash function distributes keys uniformly, average length of linked list will be $\alpha = n/m$. This ratio is called *load factor*.**

- **Average number of probes in successful, *S*, and unsuccessful searches, *U*:**

$$S \approx 1+\alpha/2, \qquad U = \alpha$$

- **Load α is typically kept small (ideally, about 1)**

- **Open hashing still works if $n > m$**

Closed hashing (Open addressing)

Keys are stored <u>inside</u> a hash table.

Key	A	FOOL	AND	HIS	MONEY	ARE	SOON	PARTED
h(*K*)	1	9	6	10	7	11	11	12

0	1	2	3	4	5	6	7	8	9	10	11	12
	A											
	A								FOOL			
	A					AND			FOOL			
	A					AND			FOOL	HIS		
	A					AND	MONEY		FOOL	HIS		
	A					AND	MONEY		FOOL	HIS	ARE	
	A					AND	MONEY		FOOL	HIS	ARE	SOON
PARTED	A					AND	MONEY		FOOL	HIS	ARE	SOON

Closed hashing (cont.)

- Does not work if $n > m$
- Avoids pointers
- Deletions are *not* straightforward
- Number of probes to find/insert/delete a key depends on load factor $\alpha = n/m$ (hash table density) and collision resolution strategy. For linear probing:

  $S = (½) (1+ 1/(1- \alpha))$ and $U = (½) (1+ 1/(1- \alpha)^2)$
- As the table gets filled (α approaches 1), number of probes in linear probing increases dramatically:

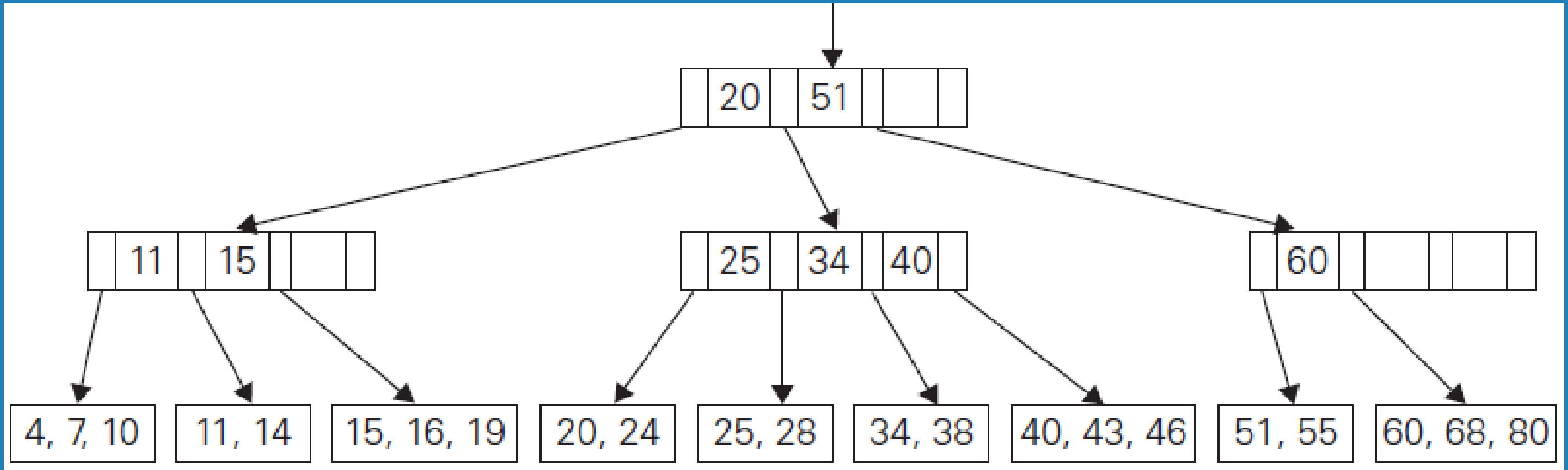
α	$\frac{1}{2}(1+\frac{1}{1-\alpha})$	$\frac{1}{2}(1+\frac{1}{(1-\alpha)^2})$
50%	1.5	2.5
75%	2.5	8.5
90%	5.5	50.5

7.4 B-Trees

- All data records (or record keys) are stored at the leaves, in increasing order of the keys
- The parental nodes are used for indexing
  - Keys are interposed with pointers to children.
  - Key left to a pointer ≤ all keys in child pointed by the pointer
    < key right to the pointer
- In addition, a B-tree of order m ≥ 2 must satisfy the following structural properties:
  - The root is either a leaf or has between 2 and *m children*.
  - Each node, except for the root and the leaves, has between *m/2 and m* children
  - The tree is balanced, i.e., all its leaves are at the same level.

B-Trees (Cont.)

- Example of a B-tree of order 4



- Search operation in B-tree
- B-tree often used for indexing large data file
  - Nodes represent disk pages
  - Minimizing the node accesses (minimizing the height) will minimizes disk accesses.

B-Trees (Cont.)

- For any B-tree of order m with n nodes and height h>0, we have the following inequality

$$n \geq 1 + \sum_{i=1}^{h-1} 2\lceil m/2 \rceil^{i-1}(\lceil m/2 \rceil - 1) + 2\lceil m/2 \rceil^{h-1}.$$

- This gives an upper bound of h

$$h \leq \lfloor \log_{\lceil m/2 \rceil} \frac{n+1}{4} \rfloor + 1.$$

- Example: for a file of 100 million records, we have

order m	50	100	250
h's upper bound	6	5	4