Chapter 3: Brute Force

A straightforward approach, usually based directly on the problem's statement and definitions of the concepts involved

Examples:

1. **Computing a^n (a > 0, n a nonnegative integer)**
2. **Computing n!**
3. **Multiplying two matrices**
4. **Searching for a key of a given value in a list**

3.1 Brute-Force Sorting Algorithm

Selection Sort **Scan the array to find its smallest element and swap it with the first element. Then, starting with the second element, scan the elements to the right of it to find the smallest among them and swap it with the second elements. Generally, on pass i ($0 \le i \le n$-2), find the smallest element in $A[i..n-1]$ and swap it with $A[i]$:**

$$A[0] \le \ . \ . \ . \ \le A[i-1] \ | \ A[i], \ . \ . \ . \ , A[min], \ . \ . \ ., A[n-1]$$

in their final positions

Example: 7 3 2 5

Analysis of Selection Sort

ALGORITHM *SelectionSort*($A[0..n-1]$)
    //Sorts a given array by selection sort
    //Input: An array $A[0..n-1]$ of orderable elements
    //Output: Array $A[0..n-1]$ sorted in ascending order
    for $i \leftarrow 0$ **to** $n-2$ **do**
        $min \leftarrow i$
        for $j \leftarrow i+1$ **to** $n-1$ **do**
            if $A[j] < A[min]$ $min \leftarrow j$
        swap $A[i]$ and $A[min]$

Time efficiency:

Space efficiency:

Stability: A sorting algorithm is called stable if it preserves the relative order of any two equal elements in its input

3.2 Brute-Force String Matching

- *pattern*: a string of *m* characters to search for
- *text*: a (longer) string of *n* characters to search in
- problem: find a substring in the text that matches the pattern

Brute-force algorithm

Step 1 Align pattern at beginning of text

Step 2 Moving from left to right, compare each character of pattern to the corresponding character in text until
  - all characters are found to match (successful search); or
  - a mismatch is detected

Step 3 While pattern is not found and the text is not yet exhausted, realign pattern one position to the right and repeat Step 2

Examples of Brute-Force String Matching

Text:
Pattern:

```
N O B O D Y _ N O T I C E D _ H I M
N O T
  N O T
    N O T
      N O T
        N O T
          N O T
            N O T
              N O T
```

Text: 1 0 0 1 0 1 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 1 1 0 1 0

Pattern:

```
0 0 1 0 1 1
  0 0 1 0 1 1
    0 0 1 0 1 1
      0 0 1 0 1 1
        0 0 1 0 1 1
```

Pseudocode and Efficiency

ALGORITHM *BruteForceStringMatch*($T[0..n-1]$, $P[0..m-1]$)

```
//Implements brute-force string matching
//Input: An array T[0..n − 1] of n characters representing a text and
//          an array P[0..m − 1] of m characters representing a pattern
//Output: The index of the first character in the text that starts a
//            matching substring or −1 if the search is unsuccessful
for i ← 0 to n − m do
      j ← 0
      while j < m and P[j] = T[i + j] do
            j ← j + 1
      if j = m return i
return −1
```

Efficiency:

Brute-Force Polynomial Evaluation

Problem: Find the value of polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0$$

at a point $x = x_0$

Brute-force algorithm

```
p ← 0.0
for i ← n downto 0 do
    power ← 1
    for j ← 1 to i do    //compute x^i
        power ← power * x
    p ← p + a[i] * power
return p
```

Efficiency:

Polynomial Evaluation: Improvement

We can do better by evaluating from right to left:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0$$

Better brute-force algorithm

```
p ← a[0]
power ← 1
for i ← 1 to n do
    power ← power * x
    p ← p + a[i] * power
return p
```

Efficiency:

3.3 Closest-Pair Problem

Find the two closest points in a set of *n* points (in the two-dimensional Cartesian plane).

<u>Brute-force algorithm</u>

Compute the distance between every pair of distinct points and return the indexes of the points for which the distance is the smallest.

Closest-Pair Brute-Force Algorithm (cont.)

ALGORITHM *BruteForceClosestPoints*(P)

//Input: A list P of n ($n \geq 2$) points $P_1 = (x_1, y_1), \ldots, P_n = (x_n, y_n)$
//Output: Indices *index*1 and *index*2 of the closest pair of points
$dmin \leftarrow \infty$
for $i \leftarrow 1$ **to** $n - 1$ **do**
  for $j \leftarrow i + 1$ **to** n **do**
    $d \leftarrow sqrt((x_i - x_j)^2 + (y_i - y_j)^2)$ //*sqrt* is the square root function
    if $d < dmin$
      $dmin \leftarrow d$; $index1 \leftarrow i$; $index2 \leftarrow j$
return *index*1, *index*2

Efficiency:

How to make it faster?

Brute-Force Strengths and Weaknesses

- Strengths
  - wide applicability
  - simplicity
  - yields reasonable algorithms for some important problems (e.g., matrix multiplication, sorting, searching, string matching)

- Weaknesses
  - rarely yields efficient algorithms
  - some brute-force algorithms are unacceptably slow
  - not as constructive as some other design techniques

3.4 Exhaustive Search

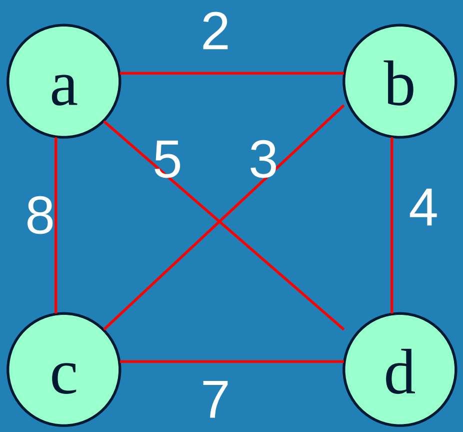
A brute force solution to a problem involving search for an element with a special property, usually among combinatorial objects such as permutations, combinations, or subsets of a set.

Method:

- generate a list of all potential solutions to the problem in a systematic manner (see algorithms in Sec. 5.4)
- evaluate potential solutions one by one, disqualifying infeasible ones and, for an optimization problem, keeping track of the best one found so far
- when search ends, announce the solution(s) found

Example 1: Traveling Salesman Problem

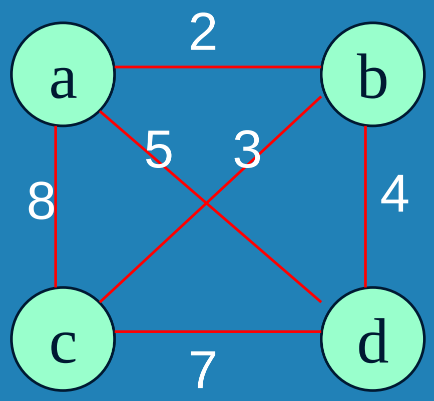
- **Given *n* cities with known distances between each pair, find the shortest tour that passes through all the cities exactly once before returning to the starting city**
- **Alternatively: Find shortest *Hamiltonian circuit* in a weighted connected graph**
- **Example:**



a cycle that passes through all the vertices of the graph exactly once

TSP by Exhaustive Search

Tour	Cost
a→b→c→d→a	2+3+7+5 = 17
a→b→d→c→a	2+4+7+8 = 21
a→c→b→d→a	8+3+4+5 = 20
a→c→d→b→a	8+7+4+2 = 21
a→d→b→c→a	5+4+3+8 = 20
a→d→c→b→a	5+7+3+2 = 17



More tours?

Less tours?

Efficiency:

Example 2: Knapsack Problem

Given *n* items:

- weights: $w_1 \quad w_2 \ \ldots \ w_n$
- values: $v_1 \quad v_2 \ \ldots \ v_n$
- a knapsack of capacity *W*

Find most valuable subset of the items that fit into the knapsack

Example: Knapsack capacity W=16

item	weight	value
1	2	$20
2	5	$30
3	10	$50
4	5	$10

Knapsack Problem by Exhaustive Search

Subset	Total weight	Total value
{1}	2	$20
{2}	5	$30
{3}	10	$50
{4}	5	$10
{1,2}	7	$50
{1,3}	12	$70
{1,4}	7	$30
{2,3}	15	$80
{2,4}	10	$40
{3,4}	15	$60
{1,2,3}	17	not feasible
{1,2,4}	12	$60
{1,3,4}	17	not feasible
{2,3,4}	20	not feasible
{1,2,3,4}	22	not feasible

Efficiency:

Example 3: The Assignment Problem

There are *n* people who need to be assigned to *n* jobs, one person per job. The cost of assigning person *i* to job *j* is C[*i*,*j*]. Find an assignment that minimizes the total cost.

	Job 1	Job 2	Job 3	Job 4
Person 1	9	2	7	8
Person 2	6	4	3	7
Person 3	5	8	1	8
Person 4	7	6	9	4

Algorithmic Plan: Generate all legitimate assignments, compute their costs, and select the cheapest one.

How many assignments are there?

Pose the problem as the one about a cost matrix:

Assignment Problem by Exhaustive Search

$$C = \begin{matrix} 9 & 2 & 7 & 8 \\ 6 & 4 & 3 & 7 \\ 5 & 8 & 1 & 8 \\ 7 & 6 & 9 & 4 \end{matrix}$$

Assignment (col.#s)	Total Cost
1, 2, 3, 4	9+4+1+4=18
1, 2, 4, 3	9+4+8+9=30
1, 3, 2, 4	9+3+8+4=24
1, 3, 4, 2	9+3+8+6=26
1, 4, 2, 3	9+7+8+9=33
1, 4, 3, 2	9+7+1+6=23
	etc.

Final Comments on Exhaustive Search

- Exhaustive-search algorithms run in a realistic amount of time only on very small instances

- In some cases, there are much better alternatives!
  - Euler circuits
  - shortest paths
  - minimum spanning tree
  - assignment problem

- In many cases, exhaustive search or its variation is the only known way to get exact solution

3.5 Graph Traversal Algorithms

Many problems require processing all graph vertices (and edges) in systematic fashion

<u>Graph traversal algorithms</u>:

- Depth-first search (DFS)
- Breadth-first search (BFS)

Depth-First Search (DFS)

- Visits graph's vertices by always moving away from last visited vertex to unvisited one, backtracks if no adjacent unvisited vertex is available.

- Uses a stack
  - a vertex is pushed onto the stack when it's reached for the first time
  - a vertex is popped off the stack when it becomes a dead end, i.e., when there is no adjacent unvisited vertex

- "Redraws" graph in tree-like fashion (with tree edges and back edges for undirected graph)

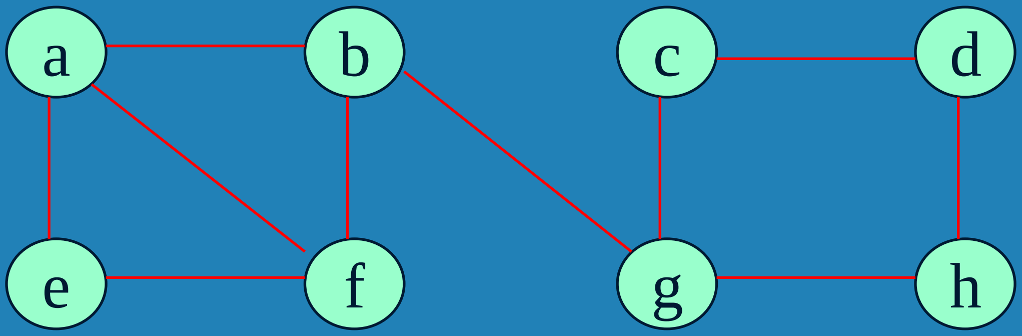
Pseudocode of DFS

ALGORITHM *DFS(G)*

```
//Implements a depth-first search traversal of a given graph
//Input: Graph G = ⟨V, E⟩
//Output: Graph G with its vertices marked with consecutive integers
//in the order they've been first encountered by the DFS traversal
mark each vertex in V with 0 as a mark of being "unvisited"
count ← 0
for each vertex v in V do
    if v is marked with 0
        dfs(v)

dfs(v)
//visits recursively all the unvisited vertices connected to vertex v by a path
//and numbers them in the order they are encountered
//via global variable count
count ← count + 1; mark v with count
for each vertex w in V adjacent to v do
    if w is marked with 0
        dfs(w)
```

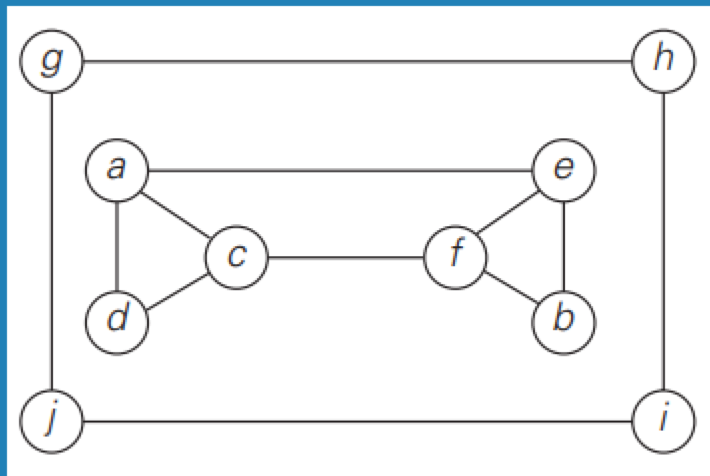
Example: DFS traversal of undirected graph



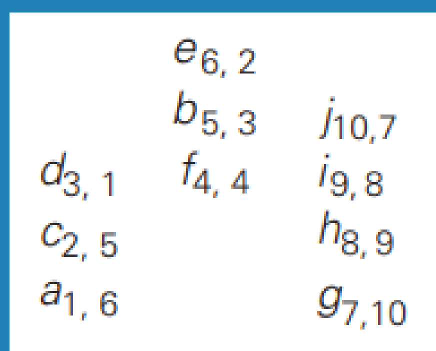
DFS traversal stack:

DFS tree:

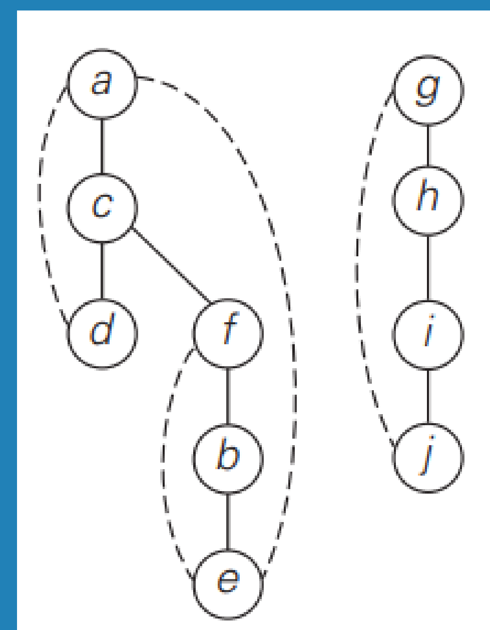
Example: DFS traversal of undirected graph



DFS traversal stack:



DFS forest:



Notes on DFS

- DFS can be implemented with graphs represented as:
  - adjacency matrices: $\Theta(V^2)$
  - adjacency lists: $\Theta(|V|+|E|)$

- Yields two distinct ordering of vertices:
  - order in which vertices are first encountered (pushed onto stack)
  - order in which vertices become dead-ends (popped off stack)

- Applications:
  - checking connectivity, finding connected components
  - checking acyclicity
  - finding articulation points
  - searching state-space of problems for solution (AI)

A vertex of a connected graph is said to be its articulation point if its removal with all edges incident to it breaks the graph into disjoint pieces.

Breadth-first search (BFS)

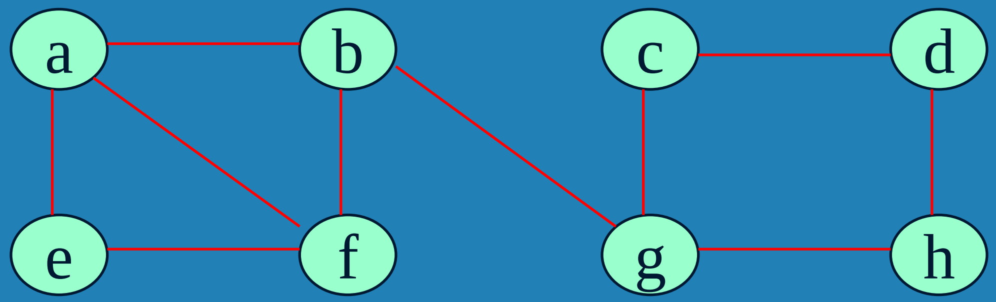
- Visits graph vertices by moving across to all the neighbors of last visited vertex
- Instead of a stack, BFS uses a queue
- Similar to level-by-level tree traversal
- “Redraws” graph in tree-like fashion (with tree edges and cross edges for undirected graph)

Pseudocode of BFS

```
ALGORITHM  BFS(G)
    //Implements a breadth-first search traversal of a given graph
    //Input: Graph G = ⟨V, E⟩
    //Output: Graph G with its vertices marked with consecutive integers
    //in the order they have been visited by the BFS traversal
    mark each vertex in V with 0 as a mark of being "unvisited"
    count ← 0
    for each vertex v in V do
        if v is marked with 0
            bfs(v)

    bfs(v)
    //visits all the unvisited vertices connected to vertex v by a path
    //and assigns them the numbers in the order they are visited
    //via global variable count
    count ← count + 1;   mark v with count and initialize a queue with v
    while the queue is not empty do
        for each vertex w in V adjacent to the front vertex do
            if w is marked with 0
                count ← count + 1;   mark w with count
                add w to the queue
        remove the front vertex from the queue
```

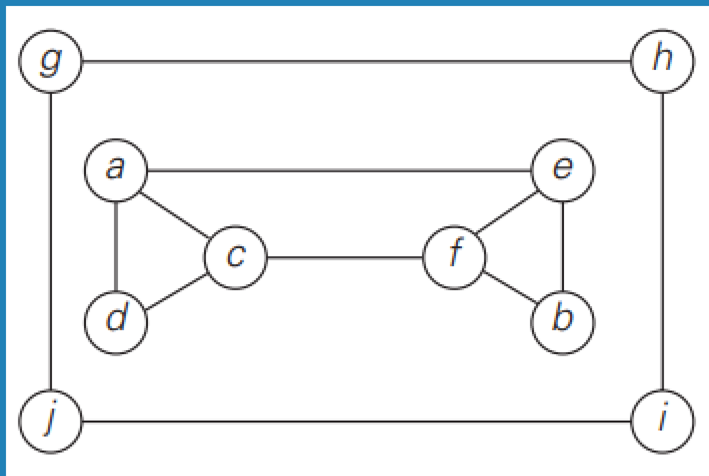
Example of BFS traversal of undirected graph



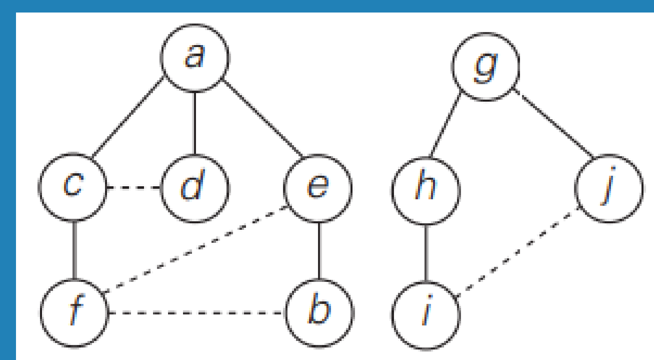
BFS traversal queue:

BFS tree:

Example: BFS traversal of undirected graph



BFS forest:



BFS traversal queue:

a_1 c_2 d_3 e_4 f_5 b_6
g_7 h_8 j_9 i_{10}

Notes on BFS

- BFS has same efficiency as DFS and can be implemented with graphs represented as:
  - adjacency matrices: $\Theta(V^2)$
  - adjacency lists: $\Theta(|V|+|E|)$
- Yields single ordering of vertices (order added/deleted from queue is the same)
- Applications: same as DFS, but can also find paths from a vertex to all other vertices with the smallest number of edges