CS 4410

# Automata, Computability, and Formal Language

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Spring 2019

# Chapter 9

## Turing Machines

1. The Standard Turing Machine
   - Definition of a Turing Machine
   - Turing Machines as Language Accepters
   - Turing Machines as Transducers
2. Combining Turing Machines for Complicated Tasks
3. Turing's Thesis

# Learning Objectives

*At the conclusion of the chapter, the student will be able to:*

- Describe the components of a standard Turing machine
- State whether an input string is accepted by a Turing machine
- Construct a Turing machine to accept a specific language
- Trace the operation of a Turing machine transducer given a sample input string
- Construct a Turing machine to compute a simple function
- State Turing’s thesis and discuss the circumstantial evidence supporting it

# Definition of a Turing Machine

Definition 9.1: A **Turing machine** M is defined by

$$M=(Q, \Sigma, \Gamma, \delta, q_0, \square, F)$$

where

Q is a finite set of **internal states**,

$\Sigma$ is the **input alphabet**,

$\Gamma$ is a finite set of symbols called the **tape alphabet**,

$\delta: Q\times\Gamma \rightarrow Q\times\Gamma\times\{L, R\}$ is the **transition function**,

$q_0 \in Q$ is the **initial state**,

$\square \in \Gamma$ is a special symbol called **blank**,

$F \subseteq Q$ is the set of **final states**

$\delta: Q\times\Gamma \rightarrow Q\times \Gamma\times\{L, R\}$

Assume $\Sigma \subseteq \Gamma - \{\square\}$

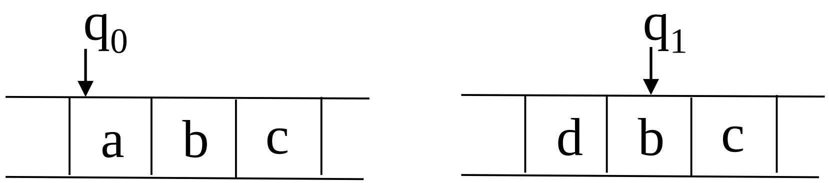
Configuration: tape symbols, state, tape head position

Halt: it reaches to a configuration for which $\delta$ is not defined

Computation: The sequence of configurations leading to a halt state.

# Examples

Example 9.1: $\delta(q_0, a) = (q_1, d, R)$



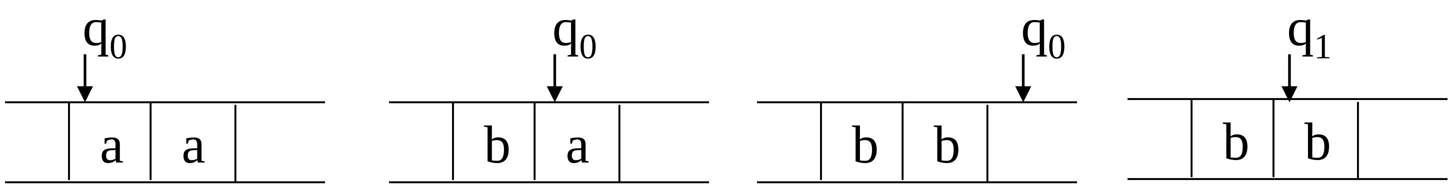

$q_0abc \vdash dq_1bc$

Example 9.2: $Q = \{q_0, q_1\}, \Sigma = \{a, b\}, \Gamma = \{a, b, \square\}, F = \{q_1\}$

$\delta(q_0, a) = (q_0, b, R)$
$\delta(q_0, b) = (q_0, b, R)$
$\delta(q_0, \square) = (q_1, \square, L)$


FIGURE 9.4

$q_0aa\square \vdash bq_0a\square \vdash bbq_0\square \vdash bq_1b\square$



# Examples

Example 9.3: $Q = \{q_0, q_1\}$, $\Sigma = \{a, b\}$, $\Gamma = \{a, b, \square\}$, $F = \{\}$

$\delta(q_0, a) = (q_1, a, R)$
$\delta(q_0, b) = (q_1, b, R)$
$\delta(q_0, \square) = (q_1, \square, R)$
$\delta(q_1, a) = (q_0, a, L)$
$\delta(q_1, b) = (q_0, b, L)$
$\delta(q_1, \square) = (q_0, \square, L)$

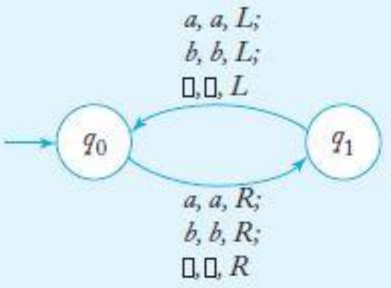


FIGURE 9.5

This machine with input string ab runs forever –in an infinite loop- with the read-write head moving alternately right and left, but making no modifications to the tape

# Standard Turing Machine

1. One tape unbounded in both directions
2. Deterministic: At most one move for each configuration
3. No special input file and No special output device

Configuration (Instantaneous description):

$x_1 q x_2$ (or $a_1 a_2 \ldots a_{k-1} q a_k a_{k+1} \ldots a_n$)

Move from one configuration to another:

$abq_1cd \vdash abeq_2d$ (if $\delta(q_1, c) = (q_2, e, R)$)

$abq_1cd \vdash aq_2bed$ (if $\delta(q_1, c) = (q_2, e, L)$)

Example 9.4, 9.5: Configurations and moves in Example 9.2

$q_0aa$ $bq_0a$ $bbq_0\square$ $bq_1b$

$q_0aa \vdash bq_0a \vdash bbq_0\square \vdash bq_1b$

# Turing Machines as Language Accepters

Definition 9.3: Let $M=(Q, \Sigma, \Gamma, \delta, q_0, \square, F)$ be a Turing machine. Then the language accepted by M is

$$L(M) = \{w \in \Sigma^+ : q_0 w \vdash^* x_1 q_f x_2 \text{ for some } g_f \in F, x_1, x_2 \in \Gamma^*\}$$

Note: input string w causes the machine to halt in a final state

Example 9.6: For $\Sigma=\{0, 1\}$, design a Turing machine M such that $L(M)=L(00^*)$

$Q=\{q_0, q_1, q_2\}$, $F=\{q_2\}$, $\Gamma =\{0, 1, \square\}$,

Example 9.7: For $\Sigma=\{0, 1\}$, design a Turing machine that accept

$$L=\{a^n b^n : n \geq 1\}$$

$Q=\{q_0, q_1, q_2 , q_3, q_4\}$, $F=\{q_4\}$, $\Gamma =\{a, b, x, y, \square\}$

Example 9.8: For $\Sigma=\{a, b, c\}$, design a Turing machine that accept

$$L=\{a^n b^n c^n : n \geq 1\}$$

$Q=\{q_0, q_1, q_2 , q_3, q_4\}$, $F=\{q_4\}$, $\Gamma =\{a, b, c, x, y, z, \square\}$

# Turing Machines as Transducers

Definition 9.3: A function f with domain D is said to be Turing-computable or just computable if there exists some Turing machine $M=(Q,\Sigma,\Gamma,\delta,q_0,\square,F)$ such that for all $w \in D$

$$q_0 w \overset{*}{\vdash} q_f f(w),\ \ q_f \in F$$

Example 9.9: Given two positive integers x and y, design a Turing machine that computes x+y.

Example 9.10: Design a Turing machine that copies strings of 1's. More precisely, find a machine that perform the computation $q_0 w \overset{*}{\vdash} q_f\, ww$ for any $w \in \{1\}^+$

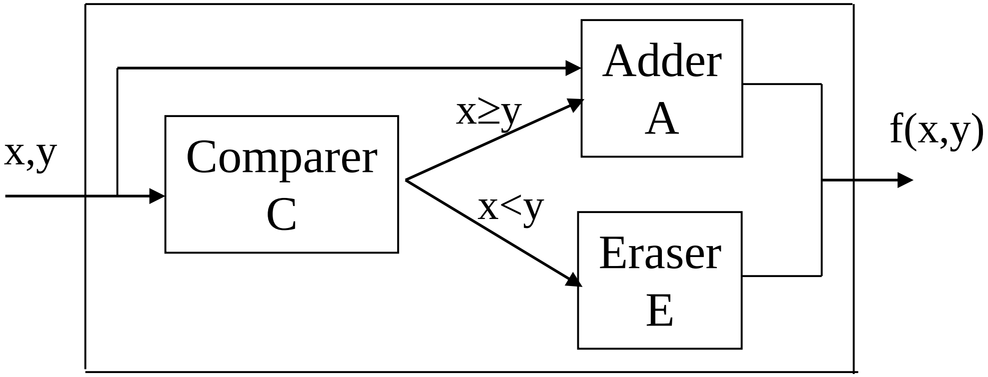
Example 9.11: Let x and y be two positive integers represented in unary notation. Construct a Turing machine that will halt in a final state $q_y$ if $x \geq y$, and that will halt in a non-final state $q_n$ if $x < y$.

$$q_0 w(x)0w(y) \overset{*}{\vdash} q_y \mathrm{w(x)0w(y)}, \qquad if\ \ x \geq y,$$
$$q_0 w(x)0w(y) \overset{*}{\vdash} q_n \mathrm{w(x)0w(y)}, \qquad if\ \ x < y.$$

# Combining Turing Machines for Complicated Tasks

Example 9.12: Design a Turing machine that computes the function

$$
\begin{aligned}
f(x,y) &= x+y, \quad if\ x \geq y, \\
&= 0, \quad\quad\ \ if\ x < y.
\end{aligned}
$$




$$q_{C,0}w(x)0w(y) \overset{*}{\vdash} q_{A,0}\mathrm{w(x)0w(y)}, \quad if\ x \geq y,$$

$$q_{C,0}w(x)0w(y) \overset{*}{\vdash} q_{E,0}\mathrm{w(x)0w(y)}, \quad if\ x < y.$$

$$q_{A,0}\mathrm{w(x)0w(y)} \overset{*}{\vdash} q_{A,f}\mathrm{w(x)w(y)0}$$

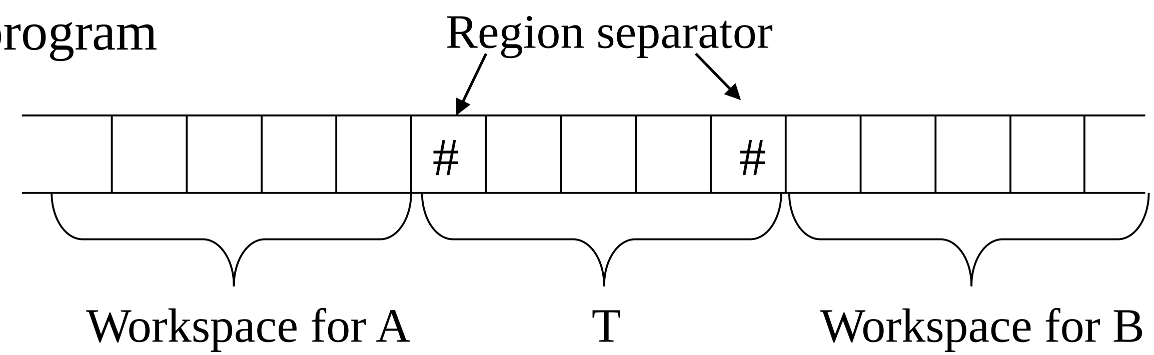
$$q_{E,0}\mathrm{w(x)0w(y)} \overset{*}{\vdash} q_{E,f}\,0$$

# Combining Turing Machines for Complicated Tasks (Cont.)

Example 9.13: Consider the instruction: *If a then* $q_j$ *else* $q_k$.

$\delta(q_i,a) = (q_{j0},a,R)$ for all $q_i \in Q$,
$\delta(q_i,b) = (q_{k0},b,R)$ for all $q_i \in Q$ and all $b \in \Gamma-\{a\}$,
$\delta(q_{j0},c) = (q_j,c,L)$ for all $c \in \Gamma$,
$\delta(q_{k0},a) = (q_k,c,L)$ for all $c \in \Gamma$.

Consider subprogram




Example 9.14: Design a Turing machine that multiplies two positive integers in unary notation.

# Turing's Thesis

Turing thesis (a hypothesis): Any computation that can be carried out by mechanical means can be performed by some Turing machine.

A computation is mechanical if and only if it can be performed by some Turing machine.

1. Anything that can be done on any existing digital computer can also be done by a Turing machine.
2. No one has yet been able to suggest a problem, solvable by what we intuitively consider an algorithm, for which a Turing machine program cannot be written.
3. Alternative models have been proposed for mechanical computation, but none of them are more powerful than the Turing machine model.

# Turing's Thesis (Cont.)

Definition 9.3: An algorithm for a function f: D→R is a Turing machine M, which given as input any $d \in D$ on its tape, eventually halts with the correct answer f(d) on its tape. Specially, we can require that

$$q_0 d \vdash_M^* q_f f(d), \quad q_f \in F$$

for all $d \in D$