CS 4410

Automata, Computability, and Formal Language

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Chapter 11: A Hierarchy of Formal Languages and Automata

1. Recursive and Recursively Enumerable Languages
   - Languages That Are Not Recursively Enumerable
   - A Language That Is Not Recursively Enumerable
   - A Language That Is Recursively Enumerable But Not Recursive
2. Unrestricted Grammars
3. Context-Sensitive Grammars and Languages
   - Context-Sensitive Languages and Linear Bounded Automata
   - Relation Between Recursive and Context-Sensitive Languages
4. The Chomsky Hierarchy

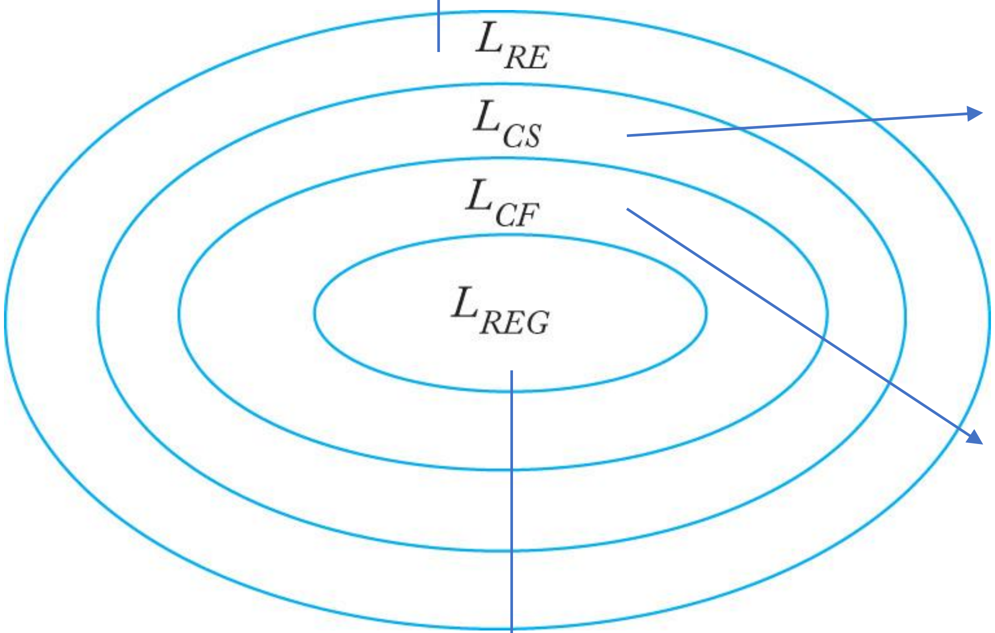
Learning Objectives

At the conclusion of the chapter, the student will be able to:

- Explain the difference between recursive and recursively enumerable languages
- Describe the type of productions in an unrestricted grammar
- Identify the types of languages generated by unrestricted grammars
- Describe the type of productions in a context sensitive grammar
- Give a sequence of derivations to generate a string using the productions in a context sensitive grammar
- Identify the types of languages generated by context-sensitive grammars
- Construct a context-sensitive grammar to generate a particular language
- Describe the structure and components of the Chomsky hierarchy

A Hierarchy of Formal Languages and Automata

Recursively enumerable language
Unrestricted Grammars, Turing Machine



Context-Sensitive Languages
Context-Sensitive grammars
Linear Bounded Automata

Context-free language
Context-free grammar, NDPA

Regular language, Regular grammar,
Regular expression, NFA, DFA

Recursive and Recursively Enumerable Languages

- A language L is *recursively enumerable* if there exists a Turing machine that accepts it (as we have previously stated, rejected strings cause the machine to either not halt or halt in a nonfinal state)
- A language L is *recursive* if there exists a Turing machine that accepts it and is guaranteed to halt on every valid input string
- In other words, a language is recursive if and only if there exists a membership algorithm for it

Languages That Are Not Recursively Enumerable

- Theorem 11.2 states that, for any nonempty alphabet, there exist languages not recursively enumerable
- One proof involves a technique called diagonalization, which can be used to show that, in a sense, there are fewer Turing Machines than there are languages
- More explicitly, Theorem 11.3 describes the existence of a recursively enumerable language whose complement is not recursively enumerable
- Furthermore, Theorem 11.5 concludes that the family of recursive languages is a proper subset of the family of recursively enumerable languages

Theorem 11.1: Let S be an infinite countable set. Then its power set 2^S is not countable

Let S = {s1, s2, s3, ...}. Then any element of 2^S can be represented by a sequence of 0's and 1's. For examples:

1 2 3 4 5 6 7 8 9

the set {s2, s3, s6} =

the set {s1, s3, s5} =

Now, suppose that 2^S were countable and 2^S ={t1, t2, t3, …}

Pick t = 0011…

Then $t \notin 2^S$

A contradiction!

So, 2^S is not countable

$\boxed{\text{Diagonalization}}$

$$
\begin{array}{cccccccc}
t_1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
t_2 & 1 & 1 & 0 & 0 & 0 & \cdots \\
t_3 & 1 & 1 & 0 & 1 & 0 & \cdots \\
t_4 & 1 & 1 & 0 & 0 & 1 & \cdots \\
\vdots & & & & & \ddots &
\end{array}
$$

Unrestricted Grammars

- An *unrestricted grammar* has essentially no restrictions on the form of its productions:
  - Any variables and terminals on the left side, in any order
  - Any variables and terminals on the right side, in any order
  - The only restriction is that λ is not allowed as the left side of a production
- A sample unrestricted grammar has productions

$$S \rightarrow S_1B$$
$$S_1 \rightarrow aS_1b$$
$$bB \rightarrow bbbB$$
$$aS_1b \rightarrow aa$$
$$B \rightarrow \lambda$$

Unrestricted Grammars and Recursively Enumerable Languages

- Theorem 11.6: Any language generated by an unrestricted grammar is recursively enumerable
- Theorem 11.7: For every recursively enumerable language L, there exists an unrestricted grammar G that generates L
- These two theorems establish the result that unrestricted grammars generate exactly the family of recursively enumerable languages, the largest family of languages that can be generated or recognized algorithmically

Context-Sensitive Grammars

- In a context-sensitive grammar, the only restriction is that, for any production, length of the right side is at least as large as the length of the left side
- Example 11.2 introduces a sample context-sensitive grammar with productions

$$S \to abc \mid aAbc$$
$$Ab \to bA$$
$$Ac \to Bbcc$$
$$bB \to Bb$$
$$aB \to aa \mid aaA$$

Derive the string aabbcc

$$\begin{aligned} S &\Rightarrow aAbc \\ &\Rightarrow abAc \\ &\Rightarrow abBbcc \\ &\Rightarrow aBbbcc \\ &\Rightarrow aabbcc \end{aligned}$$

Characteristics of Context-Sensitive Grammars

- An important characteristic of context-sensitive grammars is that they are **noncontracting**, in the sense that in any derivation, the length of successive sentential forms can never decrease
- These grammars are called context-sensitive because it is possible to specify that variables may only be replaced in certain contexts
- For instance, in the grammar of Example 11.2, variable A can only be replaced if it is followed by either b or c

$$Ab \to bA$$
$$Ac \to Bbcc$$

Context-Sensitive Languages

- A language L is context-sensitive if there is a context-sensitive grammar G, such that either $L = L(G)$ or $L = L(G) \cup \{ \lambda \}$
- The empty string is included, because by definition, a context-sensitive grammar can never generate a language containing the empty string
- As a result, it can be concluded that the family of context-free languages is a subset of the family of context-sensitive languages
- The language $\{ a^n b^n c^n : n \geq 1 \}$ is context-sensitive, since it is generated by the grammar in Example 11.2

Context-Sensitive Languages and Linear Bounded Automata

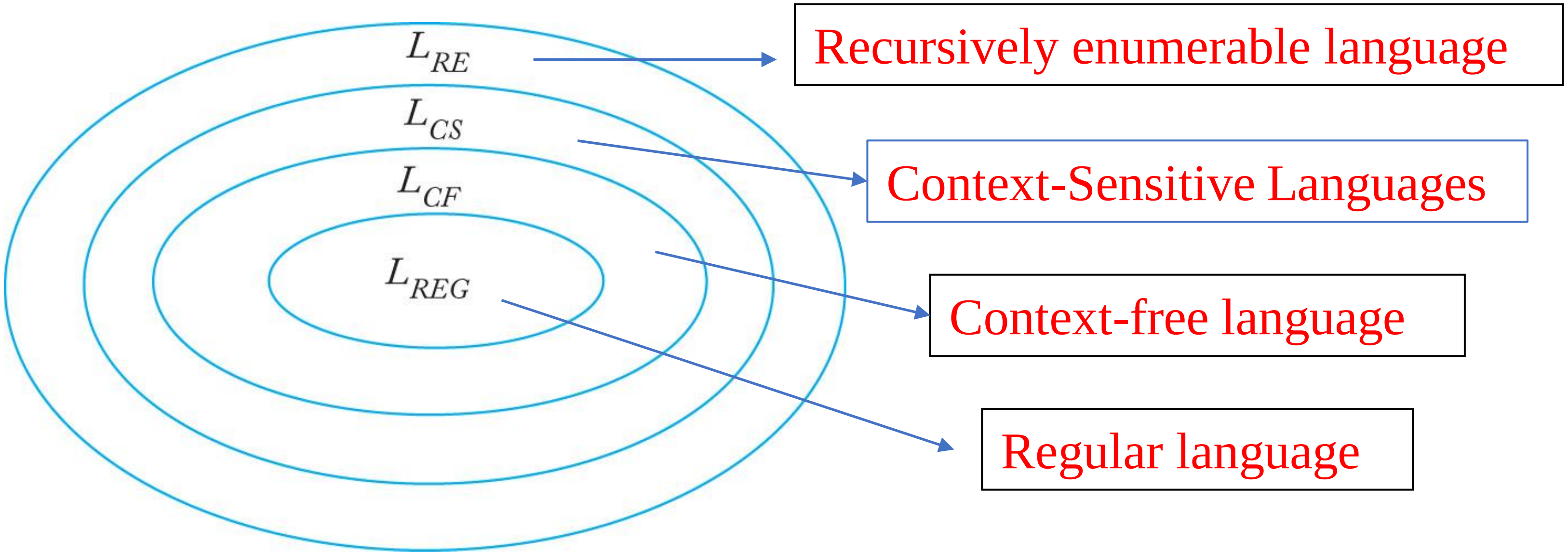
- Theorem 11.8 states that, for every context-sensitive language L not including λ, there is a linear bounded automaton that recognizes L
- Theorem 11.9 states that, if a language L is accepted by a linear bounded automaton M, then there is a context-sensitive grammar that generates L
- These two theorems establish the result that context-sensitive grammars generate exactly the family of languages accepted by linear bounded automata, the context-sensitive languages

Relationship Between Recursive and Context-Sensitive Languages

- Theorem 11.10 states that every context-sensitive language is recursive
- Theorem 11.11 maintains that some recursive languages are not context-sensitive
- These two theorems help establish a hierarchical relationship among the various classes of automata and languages:
  - Linear bounded automata are less powerful than Turing machines
  - Linear bounded automata are more powerful than pushdown automata

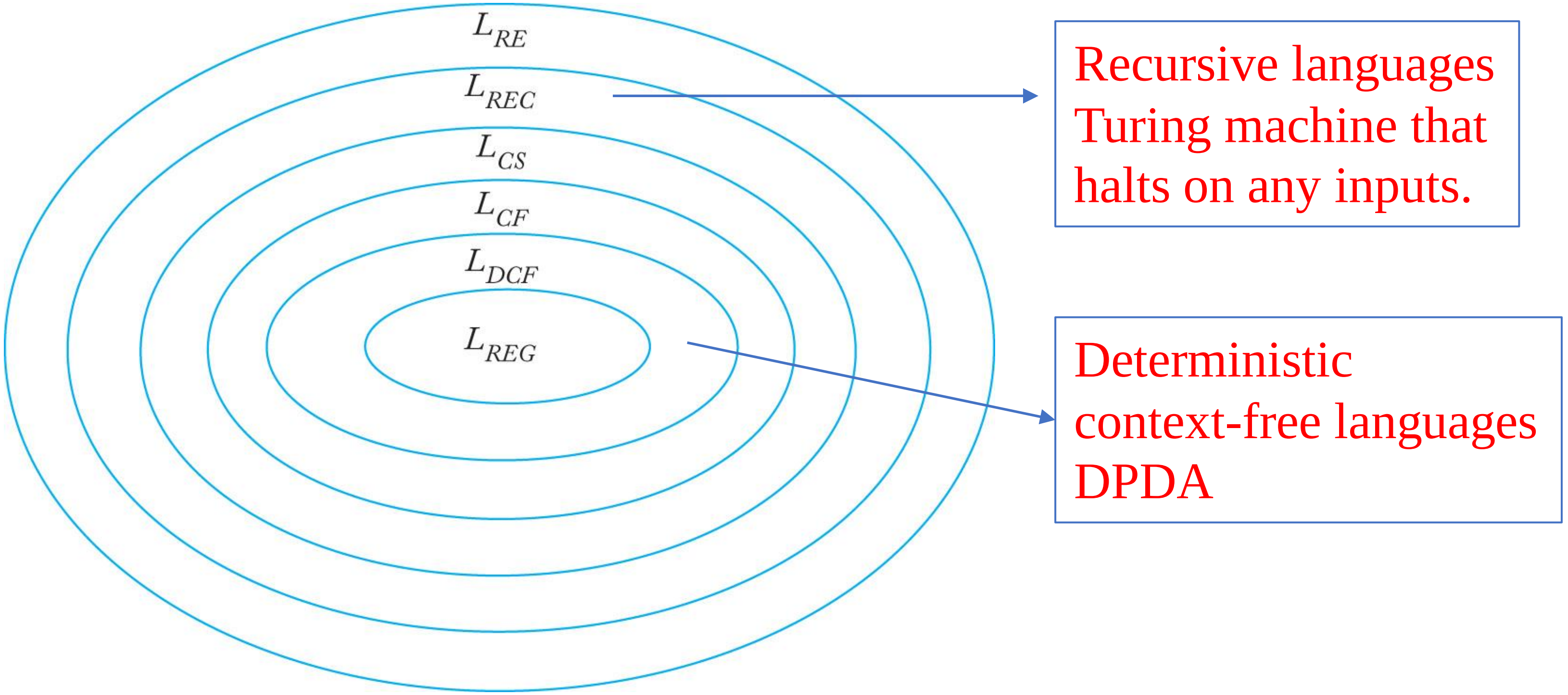
The Chomsky Hierarchy

- The linguist Noam Chomsky summarized the relationship between language families by classifying them into four language types, type 0 to type 3
- This classification, which became known as the *Chomsky Hierarchy*, is illustrated as below



An Extended Hierarchy

- We have studied additional language families and their relationships to those in the Chomsky Hierarchy
- By including deterministic context-free languages and recursive languages, we obtain the extended hierarchy as below



A Closer Look at the Family of Context-Free Languages

The following figure illustrates the relationships among various subsets of the family of context-free languages: regular (L_{REG}), linear (L_{LIN}), deterministic context-free (L_{DCF}), and nondeterministic context-free (L_{CF})

Linear languages
Linear grammars