CS 4410

Automata, Computability, and Formal Language

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1

Chapter 4 Properties of Regular Languages

- 1. Closure Properties of Regular Languages
 - Closure under Simple Set Operations
 - Closure under Other Operations
- 2. Elementary Questions about Regular Languages
- 3. Identifying Nonregular Languages
 - Using the Pigeonhole Principle
 - A pumping Lemma

Identifying Nonregular Languages

Example 4.6 (Using the pigeonhole principle): Is the language $L=\{a^nb^n: n\geq 0\}$ regular?

Using a proof by contradiction

 $a^m b^n \in L$

Suppose L is regular. Then some dfa M = (Q, {a, b}, δ , q₀, F) exists for it.

Now look at $\delta^*(q_0, a^i)$ for i = 1, 2, 3, ...

There must be some state, say q, such that

 $\delta^*(q_0, a^m) = \delta^*(q_0, a^n) = q \text{ with } m \neq n$

 $\implies \delta^*(q_0, a^m b^n) = \delta^*(\delta^*(q_0, a^m), b^n) = \delta^*(\delta^*(q_0, a^n), b^n) = \delta^*(q_0, a^n b^n)$

L is not regular

If we put n objects into m boxes (pigeonholes), and if n > m, then at least one box must have more than one item in it.

A contradiction

A pumping Lemma

Theorem 4.8 (Pumping lemma):

Let L be an infinite regular language. Then there exists some positive integer *m* such that any $w \in L$ with $|w| \ge m$ can be decomposed as

w = xyz with $|xy| \le m$, and $|y| \ge 1$,

such that

 $w_i = xy^i z \in L, i = 0, 1, 2, ...$

A pumping Lemma

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such that

$$w_i = xy^i z \in L, i = 0, 1, 2, \dots$$

 $\exists m \forall w \exists x y z \forall i (w_i \in L)$

$$m > 0$$

$$w \in L$$

$$|w| \ge m$$

$$w = xyz$$

$$|xy| \le m$$

$$|y| \ge 1$$

$$w_i = xy^i z$$

A pumping Lemma

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$$m > 0$$

$$w \in L$$

$$|w| \ge m$$

$$w = xyz$$

$$|xy| \le m$$

$$|y| \ge 1$$

$$w_i = xy^i z$$

Idea of using pumping Lemma

Show an infinite regular language L is not regular: If L violates the pumping lemma, then L is not regular.

L satisfies the pumping lemma

 $\exists m \forall w \exists x y z \forall i (w_i \in L)$

 $\neg (\exists m \forall w \exists x y z \forall i (w_i \in L)) \Leftrightarrow \\\forall m \exists w \forall x y z \exists i (w_i \notin L)$

L violates the pumping lemma

$$m > 0$$

$$w \in L$$

$$|w| \ge m$$

$$w = xyz$$

$$|xy| \le m$$

$$|y| \ge 1$$

$$w_i = xy^i z$$

Example 4.7 Using the pumping lemma to show that $L=\{a^nb^n: n\geq 0\}$ is not regular?

$$m > 0$$

$$w \in L$$

$$|w| \ge m$$

$$w = xyz$$

$$|xy| \le m$$

$$|y| \ge 1$$

$$w_i = xy^i z$$

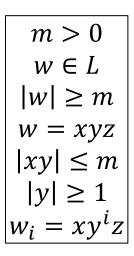
Applying the pumping lemma

$\forall m \exists w \forall x y z \exists i (w_i \notin L)$

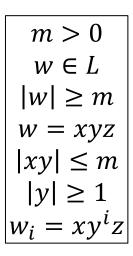
The correct argument can be visualized as a game we play against an opponent

- 1. The opponent picks m.
- 2. Given m, we pick a string w in L of length equal or greater than m.
- 3. The opponent chooses the decomposition w = xyz, subject to $|xy| \le m$, $|y| \ge 1$, in a way that makes it hard to establish a contradiction.
- 4. We try to pick *i* in such a way that the pumped string $w_i = xy^i z$ is not in L. If we can do so, we win the game.

Example 4.8: Let $\Sigma = \{a, b\}$. Show that $L = \{ww^R : w \in \Sigma^*\}$ is not regular.



Example 4.9: Let $\Sigma = \{a, b\}$. L={w \in \Sigma^*: n_a(w) < n_b(w)} is not regular.



Example 4.10: $L = \{(ab)^n a^k : n > k, k \ge 0\}$ is not regular.

$$m > 0$$

$$w \in L$$

$$|w| \ge m$$

$$w = xyz$$

$$|xy| \le m$$

$$|y| \ge 1$$

$$w_i = xy^i z$$

Example 4.11: $L = \{a^n : n \text{ is a perfect square}\}$ is not regular.

$$m > 0$$

$$w \in L$$

$$|w| \ge m$$

$$w = xyz$$

$$|xy| \le m$$

$$|y| \ge 1$$

$$w_i = xy^i z$$

Example 4.12: L={ $a^nb^kc^{n+k}$: n≥0, k≥0} is not regular.

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Example 4.13: $L = \{a^n b^k : n \neq k\}$ is not regular.

Applying the pumping lemma

- The pumping lemma says there exist an *m* as well as the decomposition *xyz*. But, we do not know what they are.
 - We cannot claim that we have reached a contradiction just because the pumping lemma is violated for some specific values of m or xyz.
- On the other hand, the pumping lemma holds for every w
 ∈ L and every and every i.
 - Therefore, if the pumping lemma is violated even for one *w* or *i*, then the language cannot be regular.

Some Common Pitfalls

- One mistake is to try using the pumping lemma to show that a language is regular. Even if you can show that no string in a language L can ever be pumped out, you cannot conclude that L is regular. The pumping lemma can only be used to prove that a language is not regular.
- Another mistake is to start (usually inadvertently) with a string not in L.
- Finally, perhaps the most common mistake is to make some assumptions about the decomposition w = xyz. The only thing we know is that y is not empty and that $|xy| \le m$;