CS 4410

Automata, Computability, and Formal Language

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Chapter 4

Properties of Regular Languages

1. Closure Properties of Regular Languages
   - Closure under Simple Set Operations
   - Closure under Other Operations
2. Elementary Questions about Regular Languages
3. Identifying Nonregular Languages
   - Using the Pigeonhole Principle
   - A pumping Lemma

Learning Objectives

At the conclusion of the chapter, the student will be able to:

- State the closure properties applicable to regular languages
- Prove that regular languages are closed under union, concatenation, star-closure, complementation, and intersection
- Prove that regular languages are closed under reversal
- Describe a membership algorithm for regular languages
- Describe an algorithm to determine if a regular language is empty, finite, or infinite
- Describe an algorithm to determine if two regular languages are equal
- Apply the pumping lemma to show that a language is not regular

Closure under Simple Set Operations

Theorem 4.1: If L, L_1 and L_2 are regular languages, then so are $L_1 \cup L_2$, $L_1 \cap L_2$, $L_1 L_2$, $\overline{L}$, and L^*. We say that the family of regular language is closed under union, intersection, concatenation, complementation, and star-closure.

Example 4.1: Show that if L_1 and L_2 are regular, so is $L_1 - L_2$.

Theorem 4.2: The family of regular languages is closed under reversal.

Closure under Other Operations

Definition 4.1: Suppose Σ and Γ are alphabets. Then a function $h: \Sigma^* \rightarrow \Gamma^*$ is called a homomorphism, if

$$h(a_1a_2\ldots a_n) = h(a_1)h(a_2)\ldots h(a_n) \qquad \text{(or } h(uv)=h(u)h(v)\text{)}$$

If L is a language on Σ, then its image is defined as

$$h(L)=\{h(w) : w\in L\}$$

Example 4.2: $\Sigma=\{a, b\}$ and $\Gamma=\{a, b, c\}$. A homomorphism h is defined as h(a)=ab and h(b)=bbc. L={aa, aba}. h(L)=?

Example 4.3: $\Sigma=\{a, b\}$ and $\Gamma=\{b, c, d\}$. A homomorphism h is defined as h(a)=dbcc and h(b)=bdc.
$r=(a+b^*)(aa)^*$ and L=L(r). Let $h(L)=L(r_1)$, r_1=?

Closure under Other Operations

Theorem 4.3: Let h be a homomorphism, if L is a regular language, then its image h(L) is also regular.

Definition 4.2: Let L_1 and L_2 be languages on the same alphabet. Then the right quotient of L1 with L2 is defined as

$$L_1/L_2=\{x : xy \in L_1 \text{ for some } y \in L_2\}$$

Example 4.4: Let $L_1=\{a^n b^m : n\geq 1, m\geq 0\}\cup\{ba\}$ and $L_2=\{b^m : m\geq 1\}$. Then L1/L2= $=\{a^n b^m : n\geq 1, m\geq 0\}$

Theorem 4.4: If L_1 and L_2 are regular, then L_1/L_2 is also regular.

Example 4.5: Let $L_1=L(a^*baa^*)$ and $L_2=L(ab^*)$. Find L_1/L_2.

Elementary Questions

Recall: What is a regular language?

Finite automaton, Regular expression, Regular grammar

Theorem 4.5: Given any regular language L on Σ and any $w \in \Sigma^*$, there exists an algorithm for determining whether or not w is in L.

Theorem 4.6: There exists an algorithm for determining whether a regular language is empty, finite, or infinite.

Theorem 4.7: Given two regular languages L_1 and L_2, there exists an algorithm for determining whether or not $L_1 = L_2$.

Identifying Nonregular Languages

Example 4.6 (Using the pigeonhole principle):

Is the language $L=\{a^n b^n: n \geq 0\}$ regular?

Theorem 4.8 (Pumping lemma):

Let L be an infinite regular language. Then there exists some positive integer m such that any $w \in L$ with $|w| \geq m$ can be decomposed as

$$w=xyz \text{ with } |xy| \leq m, \text{ and } |y| \geq 1,$$

such that

$$w_i = xy^i z \in L,\ i=0, 1, 2, \ldots$$

Example 4.7 Using the pumping lemma to show that $L=\{a^n b^n: n \geq 0\}$ is not regular?

Applying the pumping lemma (1)

- The pumping lemma says there exist an m as well as the decomposition xyz. But, we do not know what they are.
  - We cannot claim that we have reached a contradiction just because the pumping lemma is violated for some specific values of m or xyz.
- On the other hand, the pumping lemma holds for every $w \in L$ and every and every i.
  - Therefore, if the pumping lemma is violated even for one w or i, then the language cannot be regular.

Applying the pumping lemma (2)

The correct argument can be visualized as a game we play against an opponent

1. The opponent picks m.
2. Given m, we pick a string w in L of length equal or greater than m.
3. The opponent chooses the decomposition $\mathrm{w} = xyz$, subject to $|xy| \leq m$, $|y| \geq 1$, in a way that makes it hard to establish a contradiction.
4. We try to pick i in such a way that the pumped string $w_i = xy^i z$ is not in L. If we can do so, we win the game.

Identifying Nonregular Languages

Example 4.8: Let $\Sigma=\{a, b\}$. Show that $L=\{ww^R : w\in\Sigma^*\}$ is not regular.

Example 4.9: Let $\Sigma=\{a, b\}$. $L=\{w\in\Sigma^*: n_a(w)<n_b(w)\}$ is not regular.

Example 4.10: $L=\{(ab)^n a^k : n>k, k\geq 0\}$ is not regular.

Example 4.11: $L=\{a^n : n \text{ is a perfect square}\}$ is not regular.

Example 4.12: $L=\{a^n b^k c^{n+k} : n\geq 0, k\geq 0\}$ is not regular.

Example 4.13: $L=\{a^n b^k : n\neq k\}$ is not regular.

Some Common Pitfalls

- One mistake is to try using the pumping lemma to show that a language is regular. Even if you can show that no string in a language L can ever be pumped out, you cannot conclude that L is regular. The pumping lemma can only be used to prove that a language is not regular.
- Another mistake is to start (usually inadvertently) with a string not in L.
- Finally, perhaps the most common mistake is to make some assumptions about the decomposition $w = xyz$. The only thing we know is that y is not empty and that $|xy| \leq m$;