More on Solving Recurrence Relations
Math 2300, Spring 2018

Second Order Linear Recurrence Relations with Constant Coefficients

Theorem: Let p_n be ay particular solution to the recurrence relation $a_k = Aa_{k-1} + Ba_{k-2} + f(\text{k})$, ignoring initial conditions. Let q_n be the general solution to the homogeneous recurrence $a_k = Aa_{k-1} + Ba_{k-2}$ again ignoring initial conditions. Then $a_n = p_n + q_n$ is the general solution to the recurrence relation $a_k = Aa_{k-1} + Ba_{k-2} + f(k)$, where the initial conditions determine the constants in q_n.

Generating Functions:

A generating function is, roughly, a polynomial that goes on forever:

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots. + a_nx^n + \ldots.$

Definition: The generating function of a sequence $a_0, a_1, a_2, a_3, \ldots$ is the expression $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots.$

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$ generates 1, 1, 1, …

$\frac{1}{1-ax} = 1 + ax + a^2x^2 + a^3x^3 + \ldots$ generates 1, a, a^2, a^3, …

$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$ generates 1, 2, 3, 4, …

(The above theorem and definitions adapted from *Discrete Mathematics with Graph Theory* by Goodaire and Parmenter.)

Recursion Trees are often used for divide and conquer algorithms.

The Master Theorem gives bounds on recurrences:

The Master Theorem applies to recurrences of the following form:

$$T(n) = aT(n/b) + f(n)$$

where $a \geq 1$ and $b > 1$ are constants and $f(n)$ is an asymptotically positive function.

There are 3 cases:

1. If $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a} \log^k n)$ with k ≥ 0, then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.
3. If $f(n) = \Omega(n^{\log_b a + \varepsilon})$ with $\varepsilon > 0$, and $f(n)$ satisfies the regularity condition, then $T(n) = \Theta(f(n))$.
Regularity condition: $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n.

For examples of applying the Master Theorem
http://www.csanimated.com/animation.php?t=Master_theorem