One of the oldest problems in mathematics that remain unsolved is the Goldbach conjecture. In Example 3.1.5 it was shown that every even integer from 4 to 30 can be represented as a sum of two prime numbers. More than 250 years ago, Christian Goldbach (1690–1764) conjectured that every even integer greater than 2 can be so represented. Explicit computer-aided calculations have shown the conjecture to be true up to at least 10^{16}. But there is a huge chasm between 10^{16} and infinity. As pointed out by James Gleick of the *New York Times*, many other plausible conjectures in number theory have proved false. Leonhard Euler (1707–1783), for example, proposed in the eighteenth century that $a^4 + b^4 + c^4 = d^4$ had no nontrivial whole number solutions. In other words, no three perfect fourth powers add up to another perfect fourth power. For small numbers, Euler's conjecture looked good. But in 1987 a Harvard mathematician, Noam Elkies, proved it wrong. One counterexample, found by Roger Frye of Thinking Machines Corporation in a long computer search, is $95{,}800^4 + 217{,}519^4 + 414{,}560^4 = 422{,}481^4$.*

In May 2000, "to celebrate mathematics in the new millennium," the Clay Mathematics Institute of Cambridge, Massachusetts, announced that it would award prizes of $1 million each for the solutions to seven longstanding, classical mathematical questions. One of them, "P vs. NP," asks whether problems belonging to a certain class can be solved on a computer using more efficient methods than the very inefficient methods that are presently known to work for them. This question is discussed briefly at the end of Chapter 9.

Exercise Set 3.1†

In 1–3, use the definitions of even, odd, prime, and composite to justify each of your answers.

1. Assume that k is a particular integer.
   a. Is -17 an odd integer? b. Is 0 an even integer?
   c. Is $2k - 1$ odd?

2. Assume that m and n are particular integers.
   a. Is $6m + 8n$ even? b. Is $10mn + 7$ odd?
   c. If $m > n > 0$, is $m^2 - n^2$ composite?

3. Assume that r and s are particular integers.
   a. Is $4rs$ even? b. Is $6r + 4s^2 + 3$ odd?
   c. If r and s are both positive, is $r^2 + 2rs + s^2$ composite?

Prove the statements in 4–10.

4. There are integers m and n such that $m > 1$ and $n > 1$ and $\frac{1}{m} + \frac{1}{n}$ is an integer.

5. There are distinct integers m and n such that $\frac{1}{m} + \frac{1}{n}$ is an integer.

6. There are real numbers a and b such that
$$\sqrt{a+b} = \sqrt{a} + \sqrt{b}.$$

7. There is an integer $n > 5$ such that $2^n - 1$ is prime.

8. There is a real number x such that $x > 1$ and $2^x > x^{10}$.

Definition: An integer n is called a **perfect square** if, and only if, $n = k^2$ for some integer k.

9. There is a perfect square that can be written as a sum of two other perfect squares.

10. There is an integer n such that $2n^2 - 5n + 2$ is prime.

Disprove the statements in 11–13 by giving a counterexample.

11. For all real numbers a and b, if $a < b$ then $a^2 < b^2$.

12. For all integers n, if n is odd then $\frac{n-1}{2}$ is odd.

13. For all integers m and n, if $2m + n$ is odd then m and n are both odd.

In 14–16, determine whether the property is true for all integers, true for no integers, or true for some integers and false for other integers. Justify your answers.

14. $(a+b)^2 = a^2 + b^2$ 15. $3n^2 - 4n + 1$ is prime.

16. The average of any two odd integers is odd.

Prove the statements in 17 and 18 by the method of exhaustion.

17. Every positive even integer less than 26 can be expressed as a sum of three or fewer perfect squares. (For instance, $10 = 1^2 + 3^2$ and $16 = 4^2$.)

*James Gleick, "Fermat's Last Theorem Still Has Zero Solutions," *New York Times*, 17 April 1988.

†For exercises with blue numbers, solutions are given in Appendix B. The symbol ***H*** indicates that only a hint or partial solution is given. The symbol ∗ signals that an exercise is more challenging than usual.

18. For each integer n with $1 \le n \le 10$, $n^2 - n + 11$ is a prime number.

19. a. Rewrite the following theorem in the form $\forall$ ______, if ______ then ______.

b. Fill in the blanks in the proof.

Theorem: The sum of any even integer and any odd integer is odd.

Proof: Suppose m is any even integer and n is $\underline{(a)}$. By definition of even, $m = 2r$ for some $\underline{(b)}$, and by definition of odd, $n = 2s + 1$ for some integer s. By substitution and algebra, $m + n = \underline{(c)} = 2(r + s) + 1$. Since r and s are both integers, so is their sum $r + s$. Hence $m + n$ has the form $2 \cdot$ (some integer) $+ 1$, and so $\underline{(d)}$ by definition of odd.

Each of the statements in 20–23 is true. For each, write the first sentence of a proof (the "starting point") and the last sentence of a proof (the "conclusion to be shown"). Note that you do not need to understand the statements in order to be able to do these exercises.

20. For all integers m, if $m > 1$ then $0 < \frac{1}{m} < 1$.

21. For all real numbers x, if $x > 1$ then $x^2 > x$.

22. For all integers m and n, if $mn = 1$ then $m = n = 1$ or $m = n = -1$.

23. For all real numbers x, if $0 < x < 1$ then $x^2 < x$.

Prove the statements in 24–30. Follow the directions given in this section for writing proofs of universal statements.

24. The negative of any even integer is even.

25. The difference of any even integer minus any odd integer is odd.

26. The difference of any odd integer minus any even integer is odd. (*Note:* The "proof" shown in exercise 35 contains an error. Can you spot it?)

27. The sum of any two odd integers is even.

28. For all integers n, if n is odd then n^2 is odd.

29. If n is any even integer, then $(-1)^n = 1$.

30. If n is any odd integer, then $(-1)^n = -1$.

Prove that the statements in 31–33 are false.

31. There exists an integer $m \ge 3$ such that $m^2 - 1$ is prime.

32. There exists an integer n such that $6n^2 + 27$ is prime.

33. There exists an integer k such that $k \ge 4$ and $2k^2 - 5k + 2$ is prime.

Find the mistakes in the "proofs" shown in 34–38.

34. **Theorem:** For all integers k, if $k > 0$ then $k^2 + 2k + 1$ is composite.

"**Proof:** For $k = 2$, $k^2 + 2k + 1 = 2^2 + 2 \cdot 2 + 1 = 9$. But $9 = 3 \cdot 3$, and so 9 is composite. Hence the theorem is true."

35. **Theorem:** The difference between any odd integer and any even integer is odd.

"**Proof:** Suppose n is any odd integer, and m is any even integer. By definition of odd, $n = 2k + 1$ where k is an integer, and by definition of even, $m = 2k$ where k is an integer. Then $n - m = (2k + 1) - 2k = 1$. But 1 is odd. Therefore, the difference between any odd integer and any even integer is odd."

36. **Theorem:** For all integers k, if $k > 0$ then $k^2 + 2k + 1$ is composite.

"**Proof:** Suppose k is any integer such that $k > 0$. If $k^2 + 2k + 1$ is composite, then $k^2 + 2k + 1 = r \cdot s$ for some integers r and s such that $1 < r < (k^2 + 2k + 1)$ and $1 < s < (k^2 + 2k + 1)$. Since $k^2 + 2k + 1 = r \cdot s$ and both r and s are strictly between 1 and $k^2 + 2k + 1$, then $k^2 + 2k + 1$ is not prime. Hence $k^2 + 2k + 1$ is composite as was to be shown."

37. **Theorem:** The product of an even integer and an odd integer is even.

"**Proof:** Suppose m is an even integer and n is an odd integer. If $m \cdot n$ is even, then by definition of even there exists an integer r such that $m \cdot n = 2r$. Also since m is even, there exists an integer p such that $m = 2p$, and since n is odd there exists an integer q such that $n = 2q + 1$. Thus

$$m \cdot n = (2p) \cdot (2q + 1) = 2r,$$

where r is an integer. By definition of even, then, $m \cdot n$ is even, as was to be shown."

38. **Theorem:** The sum of any two even integers equals $4k$ for some integer k.

"**Proof:** Suppose m and n are any two even integers. By definition of even, $m = 2k$ for some integer k and $n = 2k$ for some integer k. By substitution, $m + n = 2k + 2k = 4k$. This is what was to be shown."

In 39–56 determine whether the statement is true or false. Justify your answer with a proof or a counterexample, as appropriate.

39. The product of any two odd integers is odd.

40. The negative of any odd integer is odd.

41. The difference of any two odd integers is odd.

42. The product of any even integer and any integer is even.

43. If a sum of two integers is even, then one of the summands is even. (In the expression $a + b$, a and b are called **summands.**)

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44. The difference of any two even integers is even.

45. The difference of any two odd integers is even.

46. For all integers n and m, if $n - m$ is even then $n^3 - m^3$ is even.

47. For all integers n, if n is prime then $(-1)^n = -1$.

48. For all integers m, if $m > 2$ then $m^2 - 4$ is composite.

49. For all integers n, $n^2 - n + 11$ is a prime number.

50. For all integers n, $4(n^2 + n + 1) - 3n^2$ is a perfect square.

51. Every positive integer can be expressed as a sum of three or fewer perfect squares.

H ✶**52.** (Two integers are **consecutive** if, and only if, one is one more than the other.) Any product of four consecutive integers is one less than a perfect square.

53. If m and n are positive integers and mn is a perfect square, then m and n are perfect squares.

54. The difference of the squares of any two consecutive integers is odd.

55. For all nonnegative real numbers a and b, $\sqrt{ab} = \sqrt{a}\sqrt{b}$. (Note that if x is a nonnegative real number, then there is a unique nonnegative real number y, denoted $\sqrt{x}$, such that $y^2 = x$.)

56. For all nonnegative real numbers a and b,

$$\sqrt{a+b} = \sqrt{a} + \sqrt{b}.$$

57. If m and n are perfect squares, then $m + n + 2\sqrt{mn}$ is also a perfect square. Why?

H ✶**58.** If p is a prime number, must $2^p - 1$ also be prime? Prove or give a counterexample.

✶ 59. If n is a nonnegative integer, must $2^{2^n} + 1$ be prime? Prove or give a counterexample.

60. When expressions of the form $(x - r)(x - s)$ are multiplied out, a quadratic polynomial is obtained. For instance, $(x - 2)(x - (-7)) = (x - 2)(x + 7) = x^2 + 5x - 14$.
    - *H* **a.** What can be said about the coefficients of the polynomial obtained by multiplying out $(x - r)(x - s)$ when both r and s are odd integers? when both r and s are even integers? when one of r and s is even and the other is odd?
    - b. It follows from part (a) that $x^2 - 1253x + 255$ cannot be written as a product of two polynomials with integer coefficients. Explain why this is so.

✶ 61. Observe that $(x - r)(x - s)(x - t)$
$$= x^3 - (r + s + t)x^2 + (rs + rt + st)x - rst.$$
    - a. Derive a result for cubic polynomials similar to the result in part (a) of exercise 60 for quadratic polynomials.
    - b. Can $x^3 + 7x^2 - 8x - 27$ be written as a product of three polynomials with integer coefficients? Explain.

3.2 Direct Proof and Counterexample II: Rational Numbers

Such, then, is the whole art of convincing. It is contained in two principles: to define all notations used, and to prove everything by replacing mentally the defined terms by their definitions. — Blaise Pascal, 1623–1662

Sums, differences, and products of integers are integers. But most quotients of integers are not integers. Quotients of integers are, however, important; they are known as *rational numbers.*

• Definition

A real number r is **rational** if, and only if, it can be expressed as a quotient of two integers with a nonzero denominator. A real number that is not rational is **irrational.** More formally, if r is a real number, then

$$r \text{ is rational } \Leftrightarrow \exists \text{ integers } a \text{ and } b \text{ such that } r = \frac{a}{b} \text{ and } b \neq 0.$$

The word *rational* contains the word *ratio,* which is another word for quotient. A rational number is a fraction or ratio of integers.

Solution Suppose a is any even integer and b is any odd integer. By property 3, b^2 is odd, and by property 1, a^2 is even. Then by property 5, $a^2 + b^2$ is odd, and because 1 is also odd, the sum $(a^2 + b^2) + 1 = a^2 + b^2 + 1$ is even by property 2. Hence, by definition of even, there exists an integer k such that $a^2 + b^2 + 1 = 2k$. Dividing both sides by 2 gives $\frac{a^2 + b^2 + 1}{2} = k$, which is an integer. Thus $\frac{a^2 + b^2 + 1}{2}$ is an integer [*as was to be shown*]. ■

A **corollary** is a statement whose truth can be immediately deduced from a theorem that has already been proved.

Example 3.2.4 The Double of a Rational Number

Derive the following as a corollary of Theorem 3.2.2.

Corollary 3.2.3

The double of a rational number is rational.

Solution The double of a number is just its sum with itself. But since the sum of any two rational numbers is rational (Theorem 3.2.2), the sum of a rational number with itself is rational. Hence the double of a rational number is rational. Here is a formal version of this argument:

Proof:

Suppose r is any rational number. Then $2r = r + r$ is a sum of two rational numbers. So, by Theorem 3.2.2, $2r$ is rational. ■

Exercise Set 3.2

The numbers in 1–7 are all rational. Write each number as a ratio of two integers.

1. $-\frac{35}{6}$ 2. 4.6037 **3.** $\frac{4}{5} + \frac{2}{9}$

4. 0.37373737...

5. 0.56565656...

6. 320.5492492492...

7. 52.4672167216721...

8. The zero product property says that if a product of two real numbers is 0, then one of the numbers must be 0.
   a. Write this property formally using quantifiers and variables.
   b. Write the contrapositive of your answer to part (a).
   c. Write an informal version (without quantifier symbols or variables) for your answer to part (b).

9. Assume that a and b are both integers and that $a \neq 0$ and $b \neq 0$. Explain why $(b - a)/(ab^2)$ must be a rational number.

10. Assume that m and n are both integers and that $n \neq 0$. Explain why $(5m + 12n)/(4n)$ must be a rational number.

11. Prove that every integer is a rational number.

12. Fill in the blanks in the following proof that the square of any rational number is rational:

Proof: Suppose that r is __(a)__. By definition of rational, $r = a/b$ for some __(b)__ with $b \neq 0$. By substitution, $r^2 =$ __(c)__ $= a^2/b^2$. Since a and b are both integers, so are the products a^2 and __(d)__. Also $b^2 \neq 0$ by the __(e)__. Hence r^2 is a ratio of two integers with a nonzero denominator, and so __(f)__ by definition of rational.

Determine which of the statements in 13–19 are true and which are false. Prove each true statement directly from the definitions, and give a counterexample for each false statement. In case the statement is false, determine whether a small change would make it true. If so, make the change and prove the new statement.

13. The product of any two rational numbers is a rational number.

H **14.** The quotient of any two rational numbers is a rational number.

15. The difference of any two rational numbers is a rational number.

16. Given any rational number r, $-r$ is also a rational number.

H **17.** If r and s are any two rational numbers with $r < s$, then $\frac{r+s}{2}$ is rational.

H **18.** For all real numbers a and b, if $a < b$ then $a < \frac{a+b}{2} < b$. (You may use the properties of inequalities in T16-T25 of Appendix A.)

19. Given any two rational numbers r and s with $r < s$, there is another rational number between r and s. (*Hint:* Use the results of exercises 17 and 18.)

Use the properties of even and odd integers that are listed in Example 3.2.3 to do exercises 20–22. Indicate which properties you use to justify your reasoning.

20. True or false? If m is any even integer and n is any odd integer, then $m^2 + 3n$ is odd. Explain.

21. True or false? If a is any odd integer, then $a^2 + a$ is even. Explain.

22. True or false? If k is any even integer and m is any odd integer, then $(k+2)^2 - (m-1)^2$ is even. Explain.

Derive the statements in 23–25 as corollaries of Theorems 3.2.1, 3.2.2, and the results of exercises 12, 13, 15, and 16.

23. For any rational numbers r and s, $2r + 3s$ is rational.

24. If r is any rational number, then $3r^2 - 2r + 4$ is rational.

25. For any rational number s, $5s^3 + 8s^2 - 7$ is rational.

26. It is a fact that if n is any nonnegative integer, then

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = \frac{1 - (1/2^{n+1})}{1 - (1/2)}.$$

(A more general form of this statement is proved in Section 4.2). Is a number of this form rational? If so, express it as a ratio of two integers.

27. Suppose a, b, c, and d are integers and $a \neq c$. Suppose also that x is a real number that satisfies the equation

$$\frac{ax+b}{cx+d} = 1.$$

Must x be rational? If so, express x as a ratio of two integers.

✶ 28. Suppose a, b, and c are integers and x, y, and z are nonzero real numbers that satisfy the following equations:

$$\frac{xy}{x+y} = a \quad \text{and} \quad \frac{xz}{x+z} = b \quad \text{and} \quad \frac{yz}{y+z} = c.$$

Is x rational? If so, express it as a ratio of two integers.

29. Prove that if one solution for a quadratic equation of the form $x^2 + bx + c = 0$ is rational (where b and c are rational), then the other solution is also rational. (Use the fact that if the solutions of the equation are r and s, then $x^2 + bx + c = (x - r)(x - s)$.)

30. Prove that if a real number c satisfies a polynomial equation of the form

$$r_3x^3 + r_2x^2 + r_1x + r_0 = 0,$$

where r_0, r_1, r_2, and r_3 are rational numbers, then c satisfies an equation of the form

$$n_3x^3 + n_2x^2 + n_1x + n_0 = 0,$$

where n_0, n_1, n_2, and n_3 are integers.

> **Definition:** A number c is called a **root** of a polynomial $p(x)$ if, and only if, $p(c) = 0$.

✶ 31. Prove that for all real numbers c, if c is a root of a polynomial with rational coefficients, then c is a root of a polynomial with integer coefficients.

In 32–36 find the mistakes in the "proofs" that the sum of any two rational numbers is a rational number.

32. "**Proof:** Let rational numbers $r = \frac{1}{4}$ and $s = \frac{1}{2}$ be given. Then $r + s = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$, which is a rational number. This is what was to be shown."

33. "**Proof:** Any two rational numbers produce a rational number when added together. So if r and s are particular but arbitrarily chosen rational numbers, then $r + s$ is rational."

34. "**Proof:** Suppose r and s are rational numbers. By definition of rational, $r = a/b$ for some integers a and b with $b \neq 0$, and $s = a/b$ for some integers a and b with $b \neq 0$. Then $r + s = a/b + a/b = 2a/b$. Let $p = 2a$. Then p is an integer since it is a product of integers. Hence $r + s = p/b$, where p and b are integers and $b \neq 0$. Thus $r + s$ is a rational number by definition of rational. This is what was to be shown."

35. "**Proof:** Suppose r and s are rational numbers. Then $r = a/b$ and $s = c/d$ for some integers a, b, c, and d with $b \neq 0$ and $d \neq 0$ (by definition of rational). Then $r + s = a/b + c/d$. But this is a sum of two fractions, which is a fraction. So $r + s$ is a rational number since a rational number is a fraction."

36. "**Proof:** Suppose r and s are rational numbers. If $r + s$ is rational, then by definition of rational $r + s = a/b$ for some integers a and b with $b \neq 0$. Also since r and s are rational, $r = i/j$ and $s = m/n$ for some integers i, j, m, and n with $j \neq 0$ and $n \neq 0$. It follows that $r + s = i/j + m/n = a/b$, which is a quotient of two integers with a nonzero denominator. Hence it is a rational number. This is what was to be shown."

The proof of the unique factorization theorem is included in Section 10.4.

Because of the unique factorization theorem, any integer $n > 1$ can be put into a *standard factored form* in which the prime factors are written in ascending order from left to right.

• Definition

Given any integer $n > 1$, the **standard factored form** of n is an expression of the form

$$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots p_k^{e_k},$$

where k is a positive integer; $p_1, p_2, \ldots, p_k$ are prime numbers; $e_1, e_2, \ldots, e_k$ are positive integers; and $p_1 < p_2 < \cdots < p_k$.

Example 3.3.10 Writing Integers in Standard Factored Form

Write 3,300 in standard factored form.

Solution First find all the factors of 3,300. Then write them in ascending order:

$$\begin{aligned} 3{,}300 &= 100 \cdot 33 = 4 \cdot 25 \cdot 3 \cdot 11 \\ &= 2 \cdot 2 \cdot 5 \cdot 5 \cdot 3 \cdot 11 = 2^2 \cdot 3^1 \cdot 5^2 \cdot 11^1. \end{aligned}$$

■

Example 3.3.11 Using Unique Factorization to Solve a Problem

Suppose m is an integer such that

$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot m = 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10.$$

Does $17 \mid m$?

Solution Since 17 is one of the prime factors of the right-hand side of the equation, it is also a prime factor of the left-hand side (by the unique factorization theorem). But 17 does not equal any prime factor of 8, 7, 6, 5, 4, 3, or 2 (because it is too large). Hence 17 must occur as one of the prime factors of m, and so $17 \mid m$. ■

Exercise Set 3.3

Give a reason for your answer in each of 1–13. Assume that all variables represent integers.

1. Is 52 divisible by 13?

2. Is 54 divisible by 18?

3. Does $5 \mid 0$?

4. Is $(3k + 1)(3k + 2)(3k + 3)$ divisible by 3?

5. Is $6m(2m + 10)$ divisible by 4?

6. Is 29 a multiple of 3?

7. Is -3 a factor of 66?

8. Is $6a(a + b)$ a multiple of $3a$?

9. Is 4 a factor of $2a \cdot 34b$?

10. Does $7 \mid 34$?

11. Does $13 \mid 73$?

12. If $n = 4k + 1$, does 8 divide $n^2 - 1$?

13. If $n = 4k + 3$, does 8 divide $n^2 - 1$?

14. Fill in the blanks in the following proof that for all integers a and b, if $a \mid b$ then $a \mid (-b)$.

Proof: Suppose a and b are any integers such that __(a)__. By definition of divisibility, $b =$ __(b)__ for some __(c)__ k. By substitution, $-b =$ __(d)__ $= a \cdot (-k)$. But $-k = (-1) \cdot k$ is an integer since -1 and k are integers. Hence, by definition of divisibility, __(e)__, as was to be shown.

Prove statements 15 and 16 directly from the definition of divisibility.

15. For all integers a, b, and c, if $a \mid b$ and $a \mid c$ then $a \mid (b + c)$.

16. For all integers a, b, and c, if $a \mid b$ and $a \mid c$ then $a \mid (b - c)$.

For each statement in 17–28, determine whether the statement is true or false. Prove the statement directly from the definitions if it is true, and give a counterexample if it is false.

17. The sum of any three consecutive integers is divisible by 3. (Two integers are **consecutive** if, and only if, one is one more than the other.)

18. The product of any two even integers is a multiple of 4.

19. A necessary condition for an integer to be divisible by 6 is that it be divisible by 2.

20. A sufficient condition for an integer to be divisible by 8 is that it be divisible by 16.

21. For all integers a, b, and c, if $a \mid b$ and $a \mid c$ then $a \mid (2b - 3c)$.

H **22.** For all integers a, b, and c, if $ab \mid c$ then $a \mid c$ and $b \mid c$.

23. For all integers a, b, and c, if a is a factor of c then ab is a factor of c.

H **24.** For all integers a, b, and c, if $a \mid (b + c)$ then $a \mid b$ or $a \mid c$.

25. For all integers a, b, and c, if $a \mid bc$ then $a \mid b$ or $a \mid c$.

26. For all integers a and b, if $a \mid b$ then $a^2 \mid b^2$.

27. For all integers a and n, if $a \mid n^2$ and $a \leq n$ then $a \mid n$.

28. For all integers a and b, if $a \mid 10b$ then $a \mid 10$ or $a \mid b$.

29. A fast-food chain has a contest in which a card with numbers on it is given to each customer who makes a purchase. If some of the numbers on the card add up to 100, then the customer wins \$100. A certain customer receives a card containing the numbers

$$72, 21, 15, 36, 69, 81, 9, 27, 42, \text{ and } 63.$$

Will the customer win \$100? Why or why not?

30. Is it possible to have a combination of nickels, dimes, and quarters that add up to \$4.72? Explain.

31. Is it possible to have 50 coins, made up of pennies, dimes, and quarters, that add up to \$3? Explain.

32. Two athletes run a circular track at a steady pace so that the first completes one round in 8 minutes and the second in 10 minutes. If they both start from the same spot at 4 P.M., when will be the first time they return to the start together?

33. It can be shown (see exercises 41–45) that an integer is divisible by 3 if, and only if, the sum of its digits is divisible by 3. An integer is divisible by 9 if, and only if, the sum of its digits is divisible by 9. An integer is divisible by 5 if, and only if, its right-most digit is a 5 or a 0. And an integer is divisible by 4 if, and only if, the number formed by its right-most two digits is divisible by 4. Check the following integers for divisibility by 3, 4, 5 and 9.
a. 637,425,403,705,125 b. 12,858,306,120,312
c. 517,924,440,926,512 d. 14,328,083,360,232

34. Use the unique factorization theorem to write the following integers in standard factored form.
a. 1176 b. 5733 c. 3675

35. Suppose that in standard factored form $a = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where k is a positive integer; $p_1, p_2, \ldots, p_k$ are prime numbers; and $e_1, e_2, \ldots, e_k$ are positive integers.
a. What is the standard factored form for a^2?
b. Find the least positive integer n such that $2^5 \cdot 3 \cdot 5^2 \cdot 7^3 \cdot n$ is a perfect square. Write the resulting product as a perfect square.
c. Find the least positive integer m such that $2^2 \cdot 3^5 \cdot 7 \cdot 11 \cdot m$ is a perfect square. Write the resulting product as a perfect square.

36. Suppose that in standard factored form $a = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where k is a positive integer; $p_1, p_2, \ldots, p_k$ are prime numbers; and $e_1, e_2, \ldots, e_k$ are positive integers.
a. What is the standard factored form for a^3?
b. Find the least positive integer k such that $2^4 \cdot 3^5 \cdot 7 \cdot 11^2 \cdot k$ is a perfect cube (i.e., equals an integer to the third power). Write the resulting product as a perfect cube.

37. **a.** If a and b are integers and $12a = 25b$, does $12 \mid b$? does $25 \mid a$? Explain.
b. If x and y are integers and $10x = 9y$, does $10 \mid y$? does $9 \mid x$? Explain.

38. How many zeros are at the end of $45^8 \cdot 88^5$? Explain how you can answer this question without actually computing the number. (*Hint:* $10 = 2 \cdot 5$.)

39. If n is an integer and $n > 1$, then $n!$ is the product of n and every other positive integer that is less than n. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.
a. Write 6! in standard factored form.
b. Write 20! in standard factored form.
c. Without computing the value of $(20!)^2$ determine how many zeros are at the end of this number when it is written in decimal form. Justify your answer.

✶40. In a certain town 2/3 of the adult men are married to 3/5 of the adult women. Assume that all marriages are monogamous (no one is married to more than one other person). Also assume that there are at least 100 adult men in the town. What is the least possible number of adult men in the town? of adult women in the town?

Definition: Given any nonnegative integer n, the **decimal representation** of n is an expression of the form

$$d_k d_{k-1} \cdots d_2 d_1 d_0,$$

where k is a nonnegative integer; $d_0, d_1, d_2, \ldots, d_k$ (called the **decimal digits** of n) are integers from 0 to 9 inclusive; $d_k \neq 0$ unless $n = 0$ and $k = 0$; and

$$n = d_k \cdot 10^k + d_{k-1} \cdot 10^{k-1} + \cdots + d_2 \cdot 10^2 + d_1 \cdot 10 + d_0.$$

(For example, $2{,}503 = 2 \cdot 10^3 + 5 \cdot 10^2 + 0 \cdot 10 + 3$.)

41. Prove that if n is any nonnegative integer whose decimal representation ends in 0, then $5 \mid n$. (*Hint:* If the decimal representation of a nonnegative integer n ends in d_0, then $n = 10m + d_0$ for some integer m.)

Note that the result of Theorem 3.4.3 can also be written, "For any odd integer n, $n^2 \mathit{ mod } 8 = 1$."

Exercise Set 3.4

For each of the values of n and d given in 1–6, find integers q and r such that $n = dq + r$ and $0 \le r < d$.

1. $n = 70, d = 9$ 2. $n = 62, d = 7$

3. $n = 36, d = 40$ 4. $n = 3, d = 11$

5. $n = -45, d = 11$ 6. $n = -27, d = 8$

Evaluate the expressions in 7–10.

7. a. 43 *div* 9 b. 43 *mod* 9

8. a. 50 *div* 7 b. 50 *mod* 7

9. a. 28 *div* 5 b. 28 *mod* 5

10. a. 30 *div* 2 b. 30 *mod* 2

11. Check the correctness of formula (3.4.1) given in Example 3.4.3 for the following values of *DayT* and N.
  a. *DayT* = 6 (Saturday) and $N = 15$
  b. *DayT* = 0 (Sunday) and $N = 7$
  c. *DayT* = 4 (Thursday) and $N = 12$

✶ 12. Justify formula (3.4.1) for general values of *DayT* and N.

13. On a Monday a friend says he will meet you again in 30 days. What day of the week will that be?

H **14.** If today is Tuesday, what day of the week will it be 1,000 days from today?

15. January 1, 2000 was a Saturday, and 2000 was a leap year. What day of the week will January 1, 2050 be?

H **16.** The / and % functions in Java operate as follows: If q and r are the integers obtained from the quotient-remainder theorem when a negative integer n is divided by a positive integer d, then n/d is $q + 1$ and $n\%d$ is $r - d$, provided that these values are assigned to an integer variable. Show that n/d and $n\%d$ satisfy one of the conclusions of the quotient-remainder theorem but not the other. To be specific, show that the equation $n = d \cdot n/d + n\%d$ is true but the condition $0 \le n\%d < d$ is false. (The functions **div** and **mod** in Pascal, / and % in C and C++, and / (or \ ) and **mod** in .NET operate similarly to / and % in Java.)

17. When an integer a is divided by 7, the remainder is 4. What is the remainder when $5a$ is divided by 7?

18. When an integer b is divided by 12, the remainder is 5. What is the remainder when $8b$ is divided by 12?

19. When an integer c is divided by 15, the remainder is 3. What is the remainder when $10c$ is divided by 15?

20. Suppose d is a positive integer and n is any integer. If $d \mid n$, what is the remainder obtained when the quotient-remainder theorem is applied to n with divisor d?

H **21.** Prove that a necessary and sufficient condition for a nonnegative integer n to be divisible by a positive integer d is that $n \mathit{ mod } d = 0$.

22. A matrix **M** has 3 rows and 4 columns.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

The 12 entries in the matrix are to be stored in *row major* form in locations 7,609 to 7,620 in a computer's memory. This means that the entries in the first row (reading left to right) are stored first, then the entries in the second row, and finally the entries in the third row.
  a. Which location will a_{22} be stored in?
  b. Write a formula (in i and j) that gives the integer n so that a_{ij} is stored in location $7{,}609 + n$.
  c. Find formulas (in n) for r and s so that a_{rs} is stored in location $7{,}609 + n$.

23. Let **M** be a matrix with m rows and n columns, and suppose that the entries of **M** are stored in a computer's memory in row major form (see exercise 22) in locations $N, N+1, N+2, \ldots, N+mn-1$. Find formulas in k for r and s so that a_{rs} is stored in location $N + k$.

24. Prove that the product of any two consecutive integers is even.

25. The result of exercise 24 suggests that the second apparent blind alley in the discussion of Example 3.4.6 might not be a blind alley after all. Write a new proof of Theorem 3.4.3 based on this observation.

26. Prove that for all integers n, $n^2 - n + 3$ is odd.

27. Show that any integer n can be written in one of the three forms

$$n = 3q \quad \text{or} \quad n = 3q + 1 \quad \text{or} \quad n = 3q + 2$$

for some integer q.

28. a. Use the quotient-remainder theorem with $d = 3$ to prove that the product of any three consecutive integers is divisible by 3.
  b. Use the *mod* notation to rewrite the result of part (a).

H **29.** Use the quotient-remainder theorem with $d = 3$ to prove that the square of any integer has the form $3k$ or $3k + 1$ for some integer k.

30. Use the quotient-remainder theorem with $d = 3$ to prove that the product of any two consecutive integers has the form $3k$ or $3k + 2$ for some integer k.

Example 3.5.6 Computing *div* and *mod*

Use the floor notation to compute 3850 *div* 17 and 3850 *mod* 17.

Solution By formula (3.5.1),

$$3850 \text{ div } 17 = \lfloor 3850/17 \rfloor = \lfloor 226.4705882\ldots \rfloor = 226$$

$$\begin{aligned} 3850 \text{ mod } 17 &= 3850 - 17 \cdot \lfloor 3850/17 \rfloor \\ &= 3850 - 17 \cdot 226 \\ &= 3850 - 3842 = 8. \end{aligned}$$

■

Exercise Set 3.5

Compute $\lfloor x \rfloor$ and $\lceil x \rceil$ for each of the values of x in 1–4.

1. 37.999 2. 17/4

3. −14.00001 4. −32/5

5. Use the floor notation to express 259 *div* 11 and 259 *mod* 11.

6. If k is an integer, what is $\lceil k \rceil$? Why?

7. If k is an integer, what is $\lceil k + \frac{1}{2} \rceil$? Why?

8. Seven pounds of raw material are needed to manufacture each unit of a certain product. Express the number of units that can be produced from n pounds of raw material using either the floor or the ceiling notation. Which notation is more appropriate?

9. Boxes, each capable of holding 36 units, are used to ship a product from the manufacturer to a wholesaler. Express the number of boxes that would be required to ship n units of the product using either the floor or the ceiling notation. Which notation is more appropriate?

10. If 0 = Sunday, 1 = Monday, 2 = Tuesday, . . . , 6 = Saturday, then January 1 of year n occurs on the day of the week given by the following formula:

$$\left(n + \left\lfloor \frac{n-1}{4} \right\rfloor - \left\lfloor \frac{n-1}{100} \right\rfloor + \left\lfloor \frac{n-1}{400} \right\rfloor\right) \text{ mod } 7.$$

a. Use this formula to find January 1 of
i. 2050 ii. 2100 iii. the year of your birth.

H **b.** Interpret the different components of this formula.

11. State a necessary and sufficient condition for the floor of a real number to equal that number.

12. Prove that if n is any even integer, then $\lfloor n/2 \rfloor = n/2$.

13. Suppose n and d are integers and $d \neq 0$. Prove each of the following.

a. If $d \mid n$, then $n = \lfloor n/d \rfloor \cdot d$.
b. If $n = \lfloor n/d \rfloor \cdot d$, then $d \mid n$.
c. Use the floor notation to state a necessary and sufficient condition for an integer n to be divisible by an integer d.

Some of the statements in 14–22 are true and some are false. Prove each true statement and find a counterexample for each false statement.

14. For all real numbers x and y, $\lfloor x - y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$.

15. For all real numbers x, $\lfloor x - 1 \rfloor = \lfloor x \rfloor - 1$.

16. For all real numbers x, $\lfloor x^2 \rfloor = \lfloor x \rfloor^2$.

H **17.** For all integers n,

$$\lfloor n/3 \rfloor = \begin{cases} n/3 & \text{if } n \text{ mod } 3 = 0 \\ (n-1)/3 & \text{if } n \text{ mod } 3 = 1 \\ (n-2)/3 & \text{if } n \text{ mod } 3 = 2 \end{cases}.$$

H **18.** For all real numbers x and y, $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$.

H **19.** For all real numbers x, $\lceil x + 1 \rceil = \lceil x \rceil + 1$.

20. For all real numbers x and y, $\lceil xy \rceil = \lceil x \rceil \cdot \lceil y \rceil$.

21. For all odd integers n, $\lceil n/2 \rceil = (n+1)/2$.

22. For all real numbers x and y, $\lceil xy \rceil = \lceil x \rceil \cdot \lfloor y \rfloor$.

Prove each of the statements in 23–29.

23. For any real number x, if x is not an integer, then $\lfloor x \rfloor + \lfloor -x \rfloor = -1$.

24. For any integer m and any real number x, if x is not an integer, then $\lfloor x \rfloor + \lfloor m - x \rfloor = m - 1$.

H **25.** For all real numbers x, $\lfloor \lfloor x/2 \rfloor /2 \rfloor = \lfloor x/4 \rfloor$.

26. For all real numbers x, if $x - \lfloor x \rfloor < 1/2$ then $\lfloor 2x \rfloor = 2\lfloor x \rfloor$.

27. For all real numbers x, if $x - \lfloor x \rfloor \geq 1/2$ then $\lfloor 2x \rfloor = 2\lfloor x \rfloor + 1$.

28. For any odd integer n,

$$\left\lfloor \frac{n^2}{4} \right\rfloor = \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right).$$

Solving problems, especially difficult problems, is rarely a straightforward process. At any stage of following the guidelines above, you might want to try the method of a previous stage again. If, for example, you fail to find a counterexample for a certain statement, your experience in trying to find it might help you decide to reattempt a direct argument rather than trying an indirect one. Psychologists who have studied problem solving have found that the most successful problem solvers are those who are flexible and willing to use a variety of approaches without getting stuck in any one of them for very long. Mathematicians sometimes work for months (or longer) on difficult problems. Don't be discouraged if some problems in this book take you quite a while to solve.

Learning the skills of proof and disproof is much like learning other skills, such as those used in swimming, tennis, or playing a musical instrument. When you first start out, you may feel bewildered by all the rules, and you may not feel confident as you attempt new things. But with practice the rules become internalized and you can use them in conjunction with all your other powers—of balance, coordination, judgment, aesthetic sense—to concentrate on winning a meet, winning a match, or playing a concert successfully.

Now that you have worked through the first six sections of this chapter, return to the idea that, above all, a proof or disproof should be a convincing argument. You need to know how direct and indirect proofs and counterexamples are structured. But to use this knowledge effectively, you must use it in conjunction with your imaginative powers, your intuition, and especially your common sense.

Exercise Set 3.6

1. Fill in the blanks in the following proof that there is no least positive real number.

Proof: [*We take the negation of the statement and suppose it to be true.*] Suppose not. That is, suppose that there is a real number x such that x is positive and __(a)__ for all positive real numbers y. [*We must deduce* __(b)__.] Consider the number $x/2$. Then __(c)__ because x is positive, and $x/2 < x$ because __(d)__. Hence __(e)__, which is a contradiction. [*Thus the supposition is false, and so there is no least positive real number.*]

2. Is $\frac{1}{0}$ an irrational number? Explain.

3. Use proof by contradiction to show that for all integers n, $3n + 2$ is not divisible by 3.

4. Use proof by contradiction to show that for all integers m, $7m + 4$ is not divisible by 7.

Carefully formulate the negations of each of the statements in 5–8. Then prove each statement by contradiction.

5. There is no greatest even integer.

6. There is no greatest negative real number.

7. There is no least positive rational number.

8. a. When asked to prove that the difference of any rational number and any irrational number is irrational, a student begins, "Suppose not. Suppose the difference of any rational number and any irrational number is rational." Comment.

b. Prove by contradiction that the difference of any rational number and any irrational numbr is irrational.

Prove each statement in 9–15 by contradiction.

9. For all real numbers x and y, if x is irrational and y is rational then $x - y$ is irrational.

10. The product of any nonzero rational number and any irrational number is irrational.

11. If a and b are rational numbers, $b \neq 0$, and r is an irrational number, then $a + br$ is irrational.

H **12.** For any integer n, $n^2 - 2$ is not divisible by 4.

H **13.** For all prime numbers a, b, and c, $a^2 + b^2 \neq c^2$.

H **14.** If a, b, and c are integers and $a^2 + b^2 = c^2$, then at least one of a and b is even.

H ✱ **15.** For all odd integers a, b, and c, if z is a solution of $ax^2 + bx + c = 0$ then z is irrational.

16. Fill in the blanks in the following proof by contraposition that for all integers n, if $5 \nmid n^2$ then $5 \nmid n$.

Proof (by contraposition): [*The contrapositive is: For all integers n, if* $5 \mid n$ *then* $5 \mid n^2$.] Suppose n is any integer such that __(a)__. [*We must show that* __(b)__.] By definition of divisibility, $n =$ __(c)__ for some integer k. By substitution, $n^2 =$ __(d)__ $= 5(5k^2)$. But $5k^2$ is an integer because it is a product of integers. Hence $n^2 = 5 \cdot$ (an integer), and so __(e)__ [*as was to be shown*].