Exercise Set 2.1*

1. A menagerie consists of seven brown dogs, two black dogs, six gray cats, ten black cats, five blue birds, six yellow birds, and one black bird. Determine which of the following statements are true and which are false.
   a. There is an animal in the menagerie that is red.
   b. Every animal in the menagerie is a bird or a mammal.
   c. Every animal in the menagerie is brown or gray or black.
   d. There is an animal in the menagerie that is neither a cat nor a dog.
   e. No animal in the menagerie is blue.
   f. There are in the menagerie a dog, a cat, and a bird that all have the same color.

2. Indicate which of the following statements are true and which are false. Justify your answers as best as you can.
   a. Every integer is a real number.
   b. 0 is a positive real number.
   c. For all real numbers r, $-r$ is a negative real number.
   d. Every real number is an integer.

3. Let $P(x)$ be the predicate "$x > 1/x$."
   a. Write $P(2)$, $P(\frac{1}{2})$, $P(-1)$, $P(-\frac{1}{2})$, and $P(-8)$, and indicate which of these statements are true and which are false.
   b. Find the truth set of $P(x)$ if the domain of x is **R**, the set of all real numbers.
   c. If the domain is the set $\mathbf{R}^+$ of all positive real numbers, what is the truth set of $P(x)$?

4. Let $Q(n)$ be the predicate "$n^2 \leq 30$."
   a. Write $Q(2)$, $Q(-2)$, $Q(7)$, and $Q(-7)$, and indicate which of these statements are true and which are false.
   b. Find the truth set of $Q(n)$ if the domain of n is **Z**, the set of all integers.
   c. If the domain is the set $\mathbf{Z}^+$ of all positive integers, what is the truth set of $Q(n)$?

5. Let $Q(x, y)$ be the predicate "If $x < y$ then $x^2 < y^2$" with domain for both x and y being the set **R** of real numbers.
   a. Explain why $Q(x, y)$ is false if $x = -2$ and $y = 1$.
   b. Give values different from those in part (a) for which $Q(x, y)$ is false.
   c. Explain why $Q(x, y)$ is true if $x = 3$ and $y = 8$.
   d. Give values different from those in part (c) for which $Q(x, y)$ is true.

6. Let $R(m, n)$ be the predicate "If m is a factor of n^2 then m is a factor of n," with domain for both m and n being the set **Z** of integers.
   a. Explain why $R(m, n)$ is false if $m = 25$ and $n = 10$.
   b. Give values different from those in part (a) for which $R(m, n)$ is false.
   c. Explain why $R(m, n)$ is true if $m = 5$ and $n = 10$.
   d. Give values different from those in part (c) for which $R(m, n)$ is true.

7. Find the truth set of each predicate.
   a. predicate: $6/d$ is an integer, domain: **Z**
   b. predicate: $6/d$ is an integer, domain: $\mathbf{Z}^+$
   c. predicate: $1 \leq x^2 \leq 4$, domain: **R**
   d. predicate: $1 \leq x^2 \leq 4$, domain: **Z**

8. Let $B(x)$ be "$-10 < x < 10$." Find the truth set of $B(x)$ for each of the following domains.
   a. **Z** b. $\mathbf{Z}^+$ c. The set of all even integers

Find counterexamples to show that the statements in 9–12 are false.

9. $\forall x \in \mathbf{R}, x > 1/x$.

10. $\forall a \in \mathbf{Z}, (a - 1)/a$ is not an integer.

11. $\forall$ positive integers m and n, $m \cdot n \geq m + n$.

12. $\forall$ real numbers x and y, $\sqrt{x + y} = \sqrt{x} + \sqrt{y}$.

13. Consider the following statement:

    $\forall$ basketball players x, x is tall.

    Which of the following are equivalent ways of expressing this statement?
   a. Every basketball player is tall.
   b. Among all the basketball players, some are tall.
   c. Some of all the tall people are basketball players.
   d. Anyone who is tall is a basketball player.
   e. All people who are basketball players are tall.
   f. Anyone who is a basketball player is a tall person.

14. Consider the following statement:

    $\exists x \in \mathbf{R}$ such that $x^2 = 2$.

    Which of the following are equivalent ways of expressing this statement?
   a. The square of each real number is 2.
   b. Some real numbers have square 2.
   c. The number x has square 2, for some real number x.
   d. If x is a real number, then $x^2 = 2$.
   e. Some real number has square 2.
   f. There is at least one real number whose square is 2.

15. Rewrite the following statements informally in at least two different ways without using variables or the symbol $\forall$ or $\exists$.
   a. $\forall$ squares x, x is a rectangle.
   b. $\exists$ a set A such that A has 16 subsets.

16. Rewrite each of the following statements in the form "$\forall$ ______ x, ______."
   a. All dinosaurs are extinct.
   b. Every real number is positive, negative, or zero.

*For exercises with blue numbers or letters, solutions are given in Appendix B. The symbol ***H*** indicates that only a hint or a partial solution is given. The symbol ✱ signals that an exercise is more challenging than usual.

c. No irrational numbers are integers.
d. No logicians are lazy.
e. The number 2,147,581,953 is not equal to the square of any integer.
f. The number -1 is not equal to the square of any real number.

17. Rewrite each of the following in the form "$\exists$ _______ x such that _______."
a. Some exercises have answers.
b. Some real numbers are rational.

18. Let D be the set of all students at your school, and let $M(s)$ be "s is a math major," let $C(s)$ be "s is a computer science student," and let $E(s)$ be "s is an engineering student." Express each of the following statements using quantifiers, variables, and the predicates $M(s)$, $C(s)$, and $E(s)$.
a. There is an engineering student who is a math major.
b. Every computer science student is an engineering student.
c. No computer science students are engineering students.
d. Some computer science students are also math majors.
e. Some computer science students are engineering students and some are not.

19. Consider the following statement:

$\forall$ integers n, if n^2 is even then n is even.

Which of the following are equivalent ways of expressing this statement?
a. All integers have even squares and are even.
b. Given any integer whose square is even, that integer is itself even.
c. For all integers, there are some whose square is even.
d. Any integer with an even square is even.
e. If the square of an integer is even, then that integer is even.
f. All even integers have even squares.

20. Rewrite the following statement informally in at least two different ways without using variables or the symbol $\forall$ or $\exists$.

$\forall$ students S, if S is in CSC 321 then S has taken MAT 140.

21. Rewrite each of the following statements in the form "$\forall$ _______ x, if _______ then _______" or "$\forall$ _______ x and y, if _______ then _______."
a. All Java programs have at least 5 lines.
b. Any valid argument with true premises has a true conclusion.
c. The sum of any two even integers is even.
d. The product of any two odd integers is odd.

22. Rewrite each of the following statements in the two forms "$\forall x$, if _______ then _______" and "$\forall$ _______ x, _______" (without an if-then).
a. The square of any even integer is even.
b. Every computer science student needs to take data structures.

23. Rewrite the following statements in the two forms "$\exists$ _______ x such that _______" and "$\exists x$ such that _______ and _______."
a. Some hatters are mad. b. Some questions are easy.

24. Consider the statement "All integers are rational numbers but some rational numbers are not integers."
a. Write this statement in the form "$\forall x$, if _______ then _______, but $\exists$ _______ x such that _______."
b. Let Ratl(x) be "x is a rational number" and Int(x) be "x is an integer." Write the given statement formally using only the symbols Ratl(x), Int(x), $\forall$, $\exists$, $\wedge$, $\vee$, $\sim$, and $\rightarrow$.

25. Refer to the picture of Tarski's world given in Example 2.1.12. Let Above(x, y) mean that x is above y (but possibly in a different column). Determine the truth or falsity of each of the following statements. Give reasons for your answers.
a. $\forall u$, Circle(u) $\rightarrow$ Gray(u).
b. $\forall u$, Gray(u) $\rightarrow$ Circle(u).
c. $\exists y$ such that Square(y) $\wedge$ Above(y, d).
d. $\exists z$ such that Triangle(z) $\wedge$ Above(f, z).

In 26–28, rewrite each statement without using quantifiers or variables. Indicate which are true and which are false, and justify your answers as best as you can.

26. Let the domain of x be the set D of objects discussed in mathematics courses, and let Real(x) be "x is a real number," Pos(x) be "x is a positive real number," Neg(x) be "x is a negative real number," and Int(x) be "x is an integer."
a. Pos(0)
b. $\forall x$, Real(x) $\wedge$ Neg(x) $\rightarrow$ Pos($-x$).
c. $\forall x$, Int(x) $\rightarrow$ Real(x).
d. $\exists x$ such that Real(x) $\wedge$ $\sim$Int(x).

27. Let the domain of x be the set of geometric figures in the plane, and let Square(x) be "x is a square" and Rect(x) be "x is a rectangle."
a. $\exists x$ such that Rect(x) $\wedge$ Square(x).
b. $\exists x$ such that Rect(x) $\wedge$ $\sim$Square(x).
c. $\forall x$, Square(x) $\rightarrow$ Rect(x).

28. Let the domain of x be the set **Z** of integers, and let Odd(x) be "x is odd," Prime(x) be "x is prime," and Square(x) be "x is a perfect square." (An integer n is said to be a **perfect square** if, and only if, it equals the square of some integer. For example, 25 is a perfect square because $25 = 5^2$.)
a. $\exists x$ such that Prime(x) $\wedge$ $\sim$Odd(x).
b. $\forall x$, Prime(x) $\rightarrow$ $\sim$Square(x).
c. $\exists x$ such that Odd(x) $\wedge$ Square(x).

H 29. In any mathematics or computer science text other than this book, find an example of a statement that is universal but is implicitly quantified. Copy the statement as it appears and rewrite it making the quantification explicit. Give a complete citation for your example, including title, author, publisher, year, and page number.

Example 2.2.6 Necessary and Sufficient Conditions

Rewrite the following statements as quantified conditional statements. Do not use the word *necessary* or *sufficient*.

a. Squareness is a sufficient condition for rectangularity.

b. Being at least 35 years old is a necessary condition for being President of the United States.

Solution

a. A formal version of the statement is

$\forall x$, if x is a square, then x is a rectangle.

Or, in informal language:

If a figure is a square, then it is a rectangle.

b. Using formal language, you could write the answer as

$\forall$ people x, if x is younger than 35, then x cannot be President of the United States.

Or, by the equivalence between a statement and its contrapositive:

$\forall$ people x, if x is President of the United States, then x is at least 35 years old. ■

Example 2.2.7 Only If

Rewrite the following as a universal conditional statement:

A product of two numbers is 0 only if one of the numbers is 0.

Solution Using informal language, you could write the answer as

If neither of two numbers is 0, then the product of the numbers is not 0.

Or, by the equivalence between a statement and its contrapositive,

If a product of two numbers is 0, then one of the numbers is 0. ■

Exercise Set 2.2

1. Which of the following is a negation for "All discrete mathematics students are athletic." More than one answer may be correct.
   a. There is a discrete mathematics student who is nonathletic.
   b. All discrete mathematics students are nonathletic.
   c. There is an athletic person who is a discrete mathematics student.
   d. No discrete mathematics students are athletic.
   e. Some discrete mathematics students are nonathletic.
   f. Some nonathletic people are not discrete mathematics students.

2. Which of the following is a negation for "All dogs are loyal"? More than one answer may be correct.
   a. All dogs are disloyal.
   b. No dogs are loyal.
   c. Some dogs are disloyal.
   d. Some dogs are loyal.
   e. There is a disloyal animal that is not a dog.
   f. There is a dog that is disloyal.
   g. No animals that are not dogs are loyal.
   h. Some animals that are not dogs are loyal.

3. Write a formal negation for each of the following statements:
   a. $\forall$ fish x, x has gills.
   b. $\forall$ computers c, c has a CPU.
   c. $\exists$ a movie m such that m is over 6 hours long.
   d. $\exists$ a band b such that b has won at least 10 Grammy awards.

4. Write an informal negation for each of the following statements:
   a. All pots have lids.
   b. All birds can fly.
   c. Some pigs can fly.
   d. Some dogs have spots.

In 5 and 6, write a formal and an informal negation for each statement in the referenced exercise.

H **5.** Section 2.1, exercise 16

H **6.** Section 2.1, exercise 17

7. Informal language is actually more complex than formal language. That is what makes the job of a systems analyst so challenging. A systems analyst works as an intermediary between a client who uses informal language and a programmer who needs precise specifications in order to produce code. For instance, the sentence "There are no orders from store *A* for item *B*" contains the words *there are.* Is the statement existential? Write an informal negation for the statement, and then write the statement formally using quantifiers and variables.

8. Consider the statement "There are no simple solutions to life's problems." Write an informal negation for the statement, and then write the statement formally using quantifiers and variables.

Write a negation for each statement in 9 and 10.

9. ∀ real numbers x, if $x > 3$ then $x^2 > 9$.

10. ∀ computer programs P, if P compiles without error messages, then P is correct.

In each of 11–14 determine whether the proposed negation is correct. If it is not, write a correct negation.

11. *Statement:* The sum of any two irrational numbers is irrational.
Proposed negation: The sum of any two irrational numbers is rational.

12. *Statement:* The product of any irrational number and any rational number is irrational.
Proposed negation: The product of any irrational number and any rational number is rational.

13. *Statement:* For all integers n, if n^2 is even then n is even.
Proposed negation: For all integers n, if n^2 is even then n is not even.

14. *Statement:* For all real numbers x_1 and x_2, if $x_1^2 = x_2^2$ then $x_1 = x_2$.
Proposed negation: For all real numbers x_1 and x_2, if $x_1^2 = x_2^2$ then $x_1 \neq x_2$.

15. Let $D = \{-48, -14, -8, 0, 1, 3, 16, 23, 26, 32, 36\}$. Determine which of the following statements are true and which are false. Provide counterexamples for those statements that are false.
a. $\forall x \in D$, if x is odd then $x > 0$.
b. $\forall x \in D$, if x is less than 0 then x is even.
c. $\forall x \in D$, if x is even then $x \leq 0$.
d. $\forall x \in D$, if the ones digit of x is 2, then the tens digit is 3 or 4.
e. $\forall x \in D$, if the ones digit of x is 6, then the tens digit is 1 or 2.

In 16 and 17, write a negation for each statement in the referenced exercise.

H **16.** Section 2.1, exercise 21

H **17.** Section 2.1, exercise 22

In 18–25, write a negation for each statement.

18. ∀ real numbers x, if $x^2 \geq 1$ then $x > 0$.

19. ∀ integers d, if $6/d$ is an integer then $d = 3$.

20. $\forall x \in \mathbf{R}$, if $x(x+1) > 0$ then $x > 0$ or $x < -1$.

21. $\forall n \in \mathbf{Z}$, if n is prime then n is odd or $n = 2$.

22. ∀ integers a, b and c, if $a - b$ is even and $b - c$ is even, then $a - c$ is even.

23. ∀ animals x, if x is a dog then x has paws and x has a tail.

24. If an integer is divisible by 2, then it is even.

25. If the square of an integer is odd, then the integer is odd.

✶26. If $P(x)$ is a predicate and the domain of x is the set of all real numbers, let R be "$\forall x \in \mathbf{Z}, P(x)$," let S be "$\forall x \in \mathbf{Q}, P(x)$," and let T be "$\forall x \in \mathbf{R}, P(x)$."
a. Find a definition for $P(x)$ (but do not use "$x \in \mathbf{Z}$") so that R is true and both S and T are false.
b. Find a definition for $P(x)$ (but do not use "$x \in \mathbf{Q}$") so that both R and S are true and T is false.

27. Consider the following string of numbers: 0204. A person claims that all the 1's in the string are to the left of all the 0's in the string. Is this true? Justify your answer. (*Hint:* Write the claim formally and write a formal negation for it. Is the negation true or false?)

28. True or false? All the occurrences of the letter u in the title of this book are lower case. Justify your answer.

In 29–36, give the contrapositive, converse, and inverse of each statement in the referenced exercise.

29. Exercise 18
30. Exercise 19
31. Exercise 20
32. Exercise 21
33. Exercise 22
34. Exercise 23
35. Exercise 24
36. Exercise 25

37. Give an example to show that a universal conditional statement is not logically equivalent to its inverse.

Rewrite each statement of 38–41 in if-then form.

38. Earning a grade of C− in this course is a sufficient condition for it to count toward graduation.

39. Being divisible by 8 is a sufficient condition for being divisible by 4.

40. Being on time each day is a necessary condition for keeping this job.

is a question asking whether block b_1 is colored blue. Prolog answers this by writing

Yes.

The statement

$$?\text{isabove}(X, w_1)$$

is a question asking for which blocks X the predicate "X is above w_1" is true. Prolog answers by giving a list of all such blocks. In this case, the answer is

$$X = b_1, X = g.$$

Note that Prolog can find the solution $X = b_1$ by merely searching the original set of given facts. However, Prolog must *infer* the solution $X = g$ from the following statements:

isabove(g, b_1),
isabove(b_1, w_1),
isabove(X, Z) if isabove(X, Y) and isabove(Y, Z).

Write the answers Prolog would give if the following questions were added to the program above.

a. ?isabove(b_2, w_1) b. ?color(w_1, X) c. ?color(X, blue)

Solution

a. The question means "Is b_2 above w_1?"; so the answer is "No."

b. The question means "For what colors X is the predicate 'w_1 is colored X' true?"; so the answer is "X = white."

c. The question means "For what blocks is the predicate 'X is colored blue' true?"; so the answer is "$X = b_1$," "$X = b_2$," and "$X = b_3$." ■

Exercise Set 2.3

1. Let C be the set of cities in the world, let N be the set of nations in the world, and let $P(c, n)$ be "c is the capital city of n." Determine the truth values of the following statements.
   a. P(Tokyo, Japan) **b.** P(Athens, Egypt)
   c. P(Paris, France) d. P(Miami, Brazil)

2. Let $G(x, y)$ be "$x^2 > y$." Indicate which of the following statements are true and which are false.
   a. $G(2, 3)$ **b.** $G(1, 1)$
   c. $G(\frac{1}{2}, \frac{1}{2})$ d. $G(-2, 2)$

3. The following statement is true: "$\forall$ nonzero numbers x, $\exists$ a real number y such that $xy = 1$." For each x given below, find a y to make the predicate "$xy = 1$" true.
   a. $x = 2$ **b.** $x = -1$ c. $x = 3/4$

4. The following statement is true: "$\forall$ real numbers x, $\exists$ an integer n such that $n > x$."* For each x given below, find an n to make the predicate "$n > x$" true.
   a. $x = 15.83$ b. $x = 10^8$ c. $x = 10^{10^{10}}$

*This is called the Archimedean principle because it was first formulated (in geometric terms) by the great Greek mathematician Archimedes of Syracuse, who lived from about 287 to 212 B.C.

The statements in exercises 5–8 refer to the Tarski world given in Example 2.3.1. Explain why each is true.

5. For all circles x there is a square y such that x and y have the same color.

6. For all squares x there is a circle y such that x and y have different colors and y is above x.

7. There is a triangle x such that for all squares y, x is above y.

8. There is a triangle x such that for all circles y, y is above x.

9. Let $D = E = \{-2, -1, 0, 1, 2\}$. Explain why the following statements are true.
   a. $\forall x$ in D, $\exists y$ in E such that $x + y = 0$.
   b. $\exists x$ in D such that $\forall y$ in E, $x + y = y$.

10. This exercise refers to Example 2.3.3. Determine whether each of the following statements is true or false.
    a. $\forall$ students S, $\exists$ a dessert D such that S chose D.
    b. $\forall$ students S, $\exists$ a salad T such that S chose T.
    c. $\exists$ a dessert D such that $\forall$ students S, S chose D.

d. $\exists$ a beverage B such that $\forall$ students D, D chose B.
e. $\exists$ an item I such that $\forall$ students S, S did not choose I.
f. $\exists$ a station Z such that $\forall$ students S, $\exists$ an item I such that S chose I from Z.

11. How could you determine the truth or falsity of the following statements for the students in your discrete mathematics class? Assume that students will respond truthfully to questions that are asked of them.
a. There is a student in this class who has dated at least one person from every residence hall at this school.
b. There is a residence hall at this school with the property that every student in this class has dated at least one person from that residence hall.
c. Every residence hall at this school has the property that if a student from this class has dated at least one person from that hall, then that student has dated at least two people from that hall.

12. Let S be the set of students at your school, let M be the set of movies that have ever been released, and let $V(s, m)$ be "student s has seen movie m." Rewrite each of the following statements without using the symbol $\forall$, the symbol $\exists$, or variables.
a. $\exists s \in S$ such that $V(s, \text{Casablanca})$.
b. $\forall s \in S$, $V(s, \text{Star Wars})$.
c. $\forall s \in S, \exists m \in M$ such that $V(s, m)$.
d. $\exists m \in M$ such that $\forall s \in S$, $V(s, m)$.
e. $\exists s \in S, \exists t \in S$, and $\exists m \in M$ such that $s \neq t$ and $V(s, m) \wedge V(t, m)$.
f. $\exists s \in S$ and $\exists t \in S$ such that $s \neq t$ and $\forall m \in M$, $V(s, m) \to V(t, m)$.

13. Let $D = E = \{-2, -1, 0, 1, 2\}$. Write negations for each of the following statements and determine which is true, the given statement or its negation.
a. $\forall x$ in $D, \exists y$ in E such that $x + y = 1$.
b. $\exists x$ in D such that $\forall y$ in E, $x + y = -y$.

In each of 14–19, (a) rewrite the statement in English without using the symbol $\forall$ or $\exists$ but expressing your answer as simply as possible, and (b) write a negation for the statement.

14. $\forall$ colors C, $\exists$ an animal A such that A is colored C.

15. $\exists$ a book b such that $\forall$ people p, p has read b.

16. $\forall$ odd integers n, $\exists$ an integer k such that $n = 2k + 1$.

17. $\forall r \in \mathbf{Q}$, $\exists$ integers a and b such that $r = a/b$.

18. $\forall x \in \mathbf{R}$, $\exists$ a real number y such that $x + y = 0$.

19. $\exists x \in \mathbf{R}$ such that for all real numbers y, $x + y = 0$.

20. Recall that reversing the order of the quantifiers in a statement with two different quantifiers may change the truth value of the statement—but it does not necessarily do so. All the statements in the pairs below refer to the Tarski world of Example 2.3.1. In each pair, the order of the quantifiers is reversed but everything else is the same. For each pair, determine whether the statements have the same or opposite truth values. Justify your answers.
a. (1) For all circles y there is a triangle x such that x and y have different colors.
(2) There is a triangle x such that for all circles y, x and y have different colors.
b. (1) For all circles y there is a square x such that x and y have the same color.
(2) There is a square x such that for all circles y, x and y have the same color.

In 21 and 22, rewrite each statement without using variables or the symbol $\forall$ or $\exists$. Indicate whether the statement is true or false.

21. a. $\forall$ real numbers x, $\exists$ a real number y such that $x + y = 0$.
b. $\exists$ a real number y such that $\forall$ real numbers x, $x + y = 0$.

22. a. $\forall$ nonzero real numbers r, $\exists$ a real number s such that $rs = 1$.
b. $\exists$ a real number s such that $\forall$ real numbers r, $rs = 1$.

23. Use the laws for negating universal and existential statements to derive the following rules:
a. $\sim(\forall x \in D(\forall y \in E(P(x, y))))$
$\equiv \exists x \in D(\exists y \in E(\sim P(x, y)))$
b. $\sim(\exists x \in D(\exists y \in E(P(x, y))))$
$\equiv \forall x \in D(\forall y \in E(\sim P(x, y)))$

Each statement in 24–27 refers to the Tarski world of Example 2.3.1. For each, (a) determine whether the statement is true or false and justify your answer, (b) write a negation for the statement (referring, if you wish, to the result in exercise 23).

24. $\forall$ circles x and $\forall$ squares y, x is above y.

25. $\forall$ circles x and $\forall$ triangles y, x is above y.

26. $\exists$ a circle x and $\exists$ a square y such that x is above y and x and y have different colors.

27. $\exists$ a circle x and $\exists$ a square y such that x is above y and x and y have the same color.

For each of the statements in 28 and 29, (a) write a new statement by interchanging the symbols $\forall$ and $\exists$, and (b) state which is true: the given statement, the version with interchanged quantifiers, neither, or both.

28. $\forall x \in \mathbf{R}, \exists y \in \mathbf{R}$ such that $x < y$.

29. $\exists x \in \mathbf{R}$ such that $\forall y \in \mathbf{R}^-$ (the set of negative real numbers), $x > y$.

30. Consider the statement "Everybody is older than somebody." Rewrite this statement in the form "$\forall$ people x, $\exists$ ______."

31. Consider the statement "Somebody is older than everybody." Rewrite this statement in the form "$\exists$ a person x such that $\forall$ ______."

In 32–38, (a) rewrite the statement formally using quantifiers and variables, and (b) write a negation for the statement.

32. Everybody loves somebody.

33. Somebody loves everybody.

34. Everybody trusts somebody.

35. Somebody trusts everybody.

36. Any even integer equals twice some integer.

37. Every action has an equal and opposite reaction.

38. There is a program that gives the correct answer to every question that is posed to it.

39. In informal speech most sentences of the form "There is ______ every ______" are intended to be understood as meaning "$\forall$ ______ $\exists$ ______," even though the existential quantifier *there is* comes before the universal quantifier *every*. Note that this interpretation applies to the following well-known sentences. Rewrite them using quantifiers and variables.
   a. There is a sucker born every minute.
   b. There is a time for every purpose under heaven.

40. Indicate which of the following statements are true and which are false. Justify your answers as best you can.
   a. $\forall x \in \mathbf{Z}^+, \exists y \in \mathbf{Z}^+$ such that $x = y + 1$.
   b. $\forall x \in \mathbf{Z}, \exists y \in \mathbf{Z}$ such that $x = y + 1$.
   c. $\exists x \in \mathbf{R}$ such that $\forall y \in \mathbf{R}, x = y + 1$.
   d. $\forall x \in \mathbf{R}^+, \exists y \in \mathbf{R}^+$ such that $xy = 1$.
   e. $\forall x \in \mathbf{R}, \exists y \in \mathbf{R}$ such that $xy = 1$.
   f. $\forall x \in \mathbf{Z}^+$ and $\forall y \in \mathbf{Z}^+, \exists z \in \mathbf{Z}^+$ such that $z = x - y$.
   g. $\forall x \in \mathbf{Z}$ and $\forall y \in \mathbf{Z}, \exists z \in \mathbf{Z}$ such that $z = x - y$.
   h. $\exists u \in \mathbf{R}^+$ such that $\forall v \in \mathbf{R}^+, uv < v$.
   i. $\forall v \in \mathbf{R}^+, \exists u \in \mathbf{R}^+$ such that $uv < v$.

41. Write the negation of the definition of limit of a sequence given in Example 2.3.7.

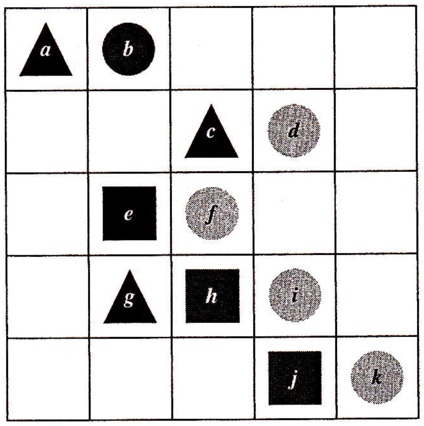
42. Write a negation for the following statement (which is the definition of $\lim_{x \to a} f(x) = f(a)$):

   For all real numbers $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for all real numbers x, if $a - \delta < x < a + \delta$ then $f(a) - \varepsilon < f(x) < f(a) + \varepsilon$.

43. The notation $\exists!$ stands for the words "there exists a unique." Thus, for instance, "$\exists!\ x$ such that x is prime and x is even" means that there is one and only one even prime number. Which of the following statements are true and which are false? Explain.
   a. $\exists!$ real number x such that $\forall$ real numbers y, $xy = y$.
   b. $\exists!$ integer x such that $1/x$ is an integer.
   c. $\forall$ real numbers x, $\exists!$ real number y such that $x + y = 0$.

✶44. Suppose that $P(x)$ is a predicate and D is the domain of x. Rewrite the statement "$\exists!\ x \in D$ such that $P(x)$" without using the symbol $\exists!$. (See exercise 43 for the meaning of $\exists!$.)

In 45–52, refer to the Tarski world given in Example 2.1.1, which is printed again here for reference. The domains of all variables consist of all the objects in the Tarski world. For each statement, (a) indicate whether the statement is true or false and justify your answer, (b) write the given statement using the formal logical notation illustrated in Example 2.3.10, and (c) write the negation of the given statement using the formal logical notation of Example 2.3.10.



45. There is a triangle x such that for all squares y, x is above y.

46. There is a triangle x such that for all circles y, x is above y.

47. For all circles x, there is a square y such that y is to the right of x.

48. For every object x, there is an object y such that if $x \neq y$ then x and y have different colors.

49. There is an object y such that for all objects x, if $x \neq y$ then x and y have different colors.

50. For all circles x and for all triangles y, x is to the right of y.

51. There is a circle x and there is a square y such that x and y have the same color.

52. There is a circle x and there is a triangle y such that x and y have the same color.

Let $P(x)$ and $Q(x)$ be predicates and suppose D is the domain of x. In 53–56, for the statement forms in each pair, determine whether (a) they have the same truth value for every choice of $P(x)$, $Q(x)$, and D, or (b) there is a choice of $P(x)$, $Q(x)$, and D for which they have opposite truth values.

53. $\forall x \in D, (P(x) \wedge Q(x))$, and
   $(\forall x \in D, P(x)) \wedge (\forall x \in D, Q(x))$

54. $\exists x \in D, (P(x) \wedge Q(x))$, and
   $(\exists x \in D, P(x)) \wedge (\exists x \in D, Q(x))$

55. $\forall x \in D, (P(x) \vee Q(x))$, and
   $(\forall x \in D, P(x)) \vee (\forall x \in D, Q(x))$

56. $\exists x \in D, (P(x) \vee Q(x))$, and
   $(\exists x \in D, P(x)) \vee (\exists x \in D, Q(x))$

Exercise Set 2.4

1. Let the following law of algebra be the first statement of an argument:

   For all real numbers a and b,
   $(a+b)^2 = a^2 + 2ab + b^2$.

   Suppose each of the following statements is, in turn, the second statement of the argument. Use universal instantiation or universal modus ponens to write the conclusion that follows in each case.

   a. $a = x$ and $b = y$ are particular real numbers.
   b. $a = f_i$ and $b = f_j$ are particular real numbers.
   c. $a = 3u$ and $b = 5v$ are particular real numbers.
   d. $a = g(r)$ and $b = g(s)$ are particular real numbers.
   e. $a = \log(t_1)$ and $b = \log(t_2)$ are particular real numbers.

Use universal instantiation or universal modus ponens to fill in valid conclusions for the arguments in 2–4.

2. If an integer n equals $2 \cdot k$ and k is an integer, then n is even.
0 equals $2 \cdot 0$ and 0 is an integer.
∴ ______________________________.

3. For all real numbers a, b, c, and d, if $b \neq 0$ and $d \neq 0$, then $a/b + c/d = (ad + bc)/bd$.
$a = 2, b = 3, c = 4$ and $d = 5$ are particular real numbers such that $b \neq 0$ and $d \neq 0$.
∴ ______________________________.

4. ∀ real numbers r, a, and b, if r is positive, then $(r^a)^b = r^{ab}$.
$r = 3, a = 1/2$, and $b = 6$ are particular real numbers such that r is positive.
∴ ______________________________.

Use universal modus tollens to fill in valid conclusions for the arguments in 5 and 6.

5. All healthy people eat an apple a day.
Adster does not eat an apple a day.
∴ ______________________________.

6. If a computer program is correct, then compilation of the program does not produce error messages.
Compilation of this program produces error messages.
∴ ______________________________.

Some of the arguments in 7–18 are valid by universal modus ponens or universal modus tollens; others are invalid and exhibit the converse or the inverse error. State which are valid and which are invalid. Justify your answers.

7. All healthy people eat an apple a day.
Keisha eats an apple a day.
∴ Keisha is a healthy person.

8. All freshmen must take writing.
Caroline is a freshman.
∴ Caroline must take writing.

9. All healthy people eat an apple a day.
Herbert is not a healthy person.
∴ Herbert does not eat an apple a day.

10. If a product of two numbers is 0, then at least one of the numbers is 0.
For a particular number x, neither $(2x+1)$ nor $(x-7)$ equals 0.
∴ The product $(2x+1)(x-7)$ is not 0.

11. All cheaters sit in the back row.
Monty sits in the back row.
∴ Monty is a cheater.

12. All honest people pay their taxes.
Darth is not honest.
∴ Darth does not pay his taxes.

13. For all students x, if x studies discrete mathematics, then x is good at logic.
Tarik studies discrete mathematics.
∴ Tarik is good at logic.

14. If compilation of a computer program produces error messages, then the program is not correct.
Compilation of this program does not produce error messages.
∴ This program is correct.

15. Any sum of two rational numbers is rational.
The sum $r + s$ is rational.
∴ The numbers r and s are both rational.

16. If a number is even, then twice that number is even.
The number $2n$ is even, for a particular number n.
∴ The particular number n is even.

17. If an infinite series converges, then the terms go to 0.
The terms of the infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ go to 0.
∴ The infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges.

18. If an infinite series converges, then its terms go to 0.
The terms of the infinite series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ do not go to 0.
∴ The infinite series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ does not converge.

19. Rewrite the statement "No good cars are cheap" in the form "$\forall x$, if $P(x)$ then $\sim Q(x)$." Indicate whether each of the following arguments is valid or invalid, and justify your answers.

   a. No good car is cheap.
   A Rimbaud is a good car.
   ∴ A Rimbaud is not cheap.

b. No good car is cheap.
A Simbaru is not cheap.
∴ A Simbaru is a good car.

c. No good car is cheap.
A VX Roadster is cheap.
∴ A VX Roadster is not good.

d. No good car is cheap.
An Omnex is not a good car.
∴ An Omnex is cheap.

20. a. Use a diagram to show that the following argument can have true premises and a false conclusion.

All dogs are carnivorous.
Aaron is not a dog.
∴ Aaron is not carnivorous.

b. What can you conclude about the validity or invalidity of the following argument form? Explain how the result from part (a) leads to this conclusion.

$\forall x$, if $P(x)$ then $Q(x)$.
$\sim P(a)$ for a particular a.
$\therefore \sim Q(a)$.

Indicate whether the arguments in 21–26 are valid or invalid. Support your answers by drawing diagrams.

21. All people are mice.
All mice are mortal.
∴ All people are mortal.

22. All discrete mathematics students can tell a valid argument from an invalid one.
All thoughtful people can tell a valid argument from an invalid one.
∴ All discrete mathematics students are thoughtful.

23. All teachers occasionally make mistakes.
No gods ever make mistakes.
∴ No teachers are gods.

24. No vegetarians eat meat.
All vegans are vegetarian.
∴ No vegans eat meat.

25. No college cafeteria food is good.
No good food is wasted.
∴ No college cafeteria food is wasted.

26. All polynomial functions are differentiable.
All differentiable functions are continuous.
∴ All polynomial functions are continuous.

27. [Adapted from Lewis Carroll.]
Nothing intelligible ever puzzles *me*.
Logic puzzles me.
∴ Logic is unintelligible.

In exercises 28–32, reorder the premises in each of the arguments to show that the conclusion follows as a valid consequence from the premises. It may be helpful to rewrite the statements in if-then form and replace some statements by their contrapositives. Exercises 28–30 refer to the kinds of Tarski worlds discussed in Example 2.1.12 and 2.3.1. Exercises 31 and 32 are adapted from *Symbolic Logic* by Lewis Carroll.*

28. 1. Every object that is to the right of all the blue objects is above all the triangles.
2. If an object is a circle, then it is to the right of all the blue objects.
3. If an object is not a circle, then it is not gray.
∴ All the gray objects are above all the triangles.

29. 1. All the objects that are to the right of all the triangles are above all the circles.
2. If an object is not above all the black objects, then it is not a square.
3. All the objects that are above all the black objects are to the right of all the triangles.
∴ All the squares are above all the circles.

30. 1. If an object is above all the triangles, then it is above all the blue objects.
2. If an object is not above all the gray objects, then it is not a square.
3. Every black object is a square.
4. Every object that is above all the gray objects is above all the triangles.
∴ If an object is black, then it is above all the blue objects.

31. 1. I trust every animal that belongs to me.
2. Dogs gnaw bones.
3. I admit no animals into my study unless they will beg when told to do so.
4. All the animals in the yard are mine.
5. I admit every animal that I trust into my study.
6. The only animals that are really willing to beg when told to do so are dogs.
∴ All the animals in the yard gnaw bones.

32. 1. When I work a logic example without grumbling, you may be sure it is one I understand.
2. The arguments in these examples are not arranged in regular order like the ones I am used to.
3. No easy examples make my head ache.
4. I can't understand examples if the arguments are not arranged in regular order like the ones I am used to.
5. I never grumble at an example unless it gives me a headache.
∴ These examples are not easy.

*Lewis Carroll, *Symbolic Logic* (New York: Dover, 1958), pp. 118, 120, 123.