Exercise Set 10.1*

1. Let $A = \{2, 3, 4\}$ and $B = \{6, 8, 10\}$ and define a binary relation R from A to B as follows:

   For all $(x, y) \in A \times B$, $\quad (x, y) \in R \quad \Leftrightarrow \quad x \mid y$.

   a. Is $4\ R\ 6$? Is $4\ R\ 8$? Is $(3, 8) \in R$? Is $(2, 10) \in R$?
   b. Write R as a set of ordered pairs.

2. Let $C = \{2, 3, 4, 5\}$ and $D = \{3, 4\}$ and define a binary relation S from C to D as follows:

   For all $(x, y) \in C \times D$, $\quad (x, y) \in S \quad \Leftrightarrow \quad x \geq y$.

   a. Is $2\ S\ 4$? Is $4\ S\ 3$? Is $(4, 4) \in S$? Is $(3, 2) \in S$?
   b. Write S as a set of ordered pairs.

3. As in Example 10.1.2, the **congruence modulo 2** relation E is defined from **Z** to **Z** as follows:

   For all integers m and n, $\quad m\ E\ n \quad \Leftrightarrow \quad m - n$ is even.

   a. Is $0\ E\ 0$? Is $5\ E\ 2$? Is $(6, 6) \in E$? Is $(-1, 7) \in E$?
   b. Prove that for any even integer n, $n\ E\ 0$.

H **4.** Prove that for all integers m and n, $m - n$ is even if, and only if, both m and n are even or both m and n are odd.

5. The **congruence modulo 3** relation, T, is defined from **Z** to **Z** as follows:

   For all integers m and n, $\quad m\ T\ n \quad \Leftrightarrow \quad 3 \mid (m - n)$.

   a. Is $10\ T\ 1$? Is $1\ T\ 10$? Is $(2, 2) \in T$? Is $(8, 1) \in T$?
   b. List five integers n such that $n\ T\ 0$.
   c. List five integers n such that $n\ T\ 1$.
   d. List five integers n such that $n\ T\ 2$.
   H **e.** Make and prove a conjecture about which integers are related by T to 0, which integers are related by T to 1, and which integers are related by T to 2.

6. Define a binary relation S from **R** to **R** as follows:

   For all $(x, y) \in \mathbf{R} \times \mathbf{R}$, $\quad x\ S\ y \quad \Leftrightarrow \quad x \geq y$.

   a. Is $(2, 1) \in S$? Is $(2, 2) \in S$? Is $2\ S\ 3$? Is $(-1)\ S\ (-2)$?
   b. Draw the graph of S in the Cartesian plane.

7. Define a binary relation R from **R** to **R** as follows:

   For all $(x, y) \in \mathbf{R} \times \mathbf{R}$, $\quad x\ R\ y \quad \Leftrightarrow \quad y = x^2$.

   a. Is $(2, 4) \in R$? Is $(4, 2) \in R$? Is $(-3)\ R\ 9$? Is $9\ R\ (-3)$?
   b. Draw the graph of R in the Cartesian plane.

8. Define a binary relation P on **Z** as follows:

   For all $m, n \in \mathbf{Z}$,

   $m\ P\ n \quad \Leftrightarrow \quad m$ and n have a common prime factor.

   a. Is $15\ P\ 25$? **b.** $22\ P\ 27$?
   c. Is $0\ P\ 5$? d. Is $8\ P\ 8$?

9. Let $X = \{a, b, c\}$. Recall that $\mathscr{P}(X)$ is the power set of X. Define a binary relation $\mathscr{R}$ on $\mathscr{P}(X)$ as follows:

   For all $A, B \in \mathscr{P}(X)$,

   $A\ \mathscr{R}\ B \quad \Leftrightarrow \quad A$ has the same number of elements as B.

   a. Is $\{a, b\}\ \mathscr{R}\ \{b, c\}$? b. Is $\{a\}\ \mathscr{R}\ \{a, b\}$?
   c. Is $\{c\}\ \mathscr{R}\ \{b\}$?

10. Let $X = \{a, b, c\}$. Define a binary relation $\mathscr{J}$ on $\mathscr{P}(X)$ as follows:

    For all $A, B \in \mathscr{P}(X)$, $\quad A\ \mathscr{J}\ B \quad \Leftrightarrow \quad A \cap B \neq \emptyset$.

    a. Is $\{a\}\ \mathscr{J}\ \{c\}$? b. Is $\{a, b\}\ \mathscr{J}\ \{b, c\}$?
    c. Is $\{a, b\}\ \mathscr{J}\ \{a, b, c\}$?

11. Let S be the set of all strings of a's and b's of length 4. Define a relation R on S as follows:

    For all $s, t \in S$,

    $s\ R\ t \quad \Leftrightarrow \quad s$ has the same first two characters as t.

    a. Is $abaa\ R\ abba$? b. Is $aabb\ R\ bbaa$?
    c. Is $aaaa\ R\ aaab$?

H **12.** Let $A = \{4, 5, 6\}$ and $B = \{5, 6, 7\}$ and define binary relations R, S, and T from A to B as follows:

    For all $(x, y) \in A \times B$, $\quad (x, y) \in R \quad \Leftrightarrow \quad x \geq y$.
    For all $(x, y) \in A \times B$, $\quad x\ S\ y \quad \Leftrightarrow \quad 2 \mid (x - y)$.
    $T = \{(4, 7), (6, 5), (6, 7)\}$.

    a. Draw arrow diagrams for R, S, and T.
    b. Indicate whether any of the relations R, S, and T are functions.

13. a. Find all binary relations from $\{0,1\}$ to $\{1\}$.
    b. Find all functions from $\{0,1\}$ to $\{1\}$.
    c. What fraction of the binary relations from $\{0, 1\}$ to $\{1\}$ are functions?

14. Find four binary relations from $\{a, b\}$ to $\{x, y\}$ that are not functions from $\{a, b\}$ to $\{x, y\}$.

H **15.** Suppose A is a set with m elements and B is a set with n elements.
    a. How many binary relations are there from A to B? Explain.
    b. How many functions are there from A to B? Explain.
    c. What fraction of the binary relations from A to B are functions?

16. Define a binary relation P from **R** to **R** as follows:

    For all real numbers x and y,

    $(x, y) \in P \quad \Leftrightarrow \quad x = y^2$.

    Is P a function? Explain.

*For exercises with blue numbers or letters, solutions are given in Appendix B. The symbol ***H*** indicates that only a hint or a partial solution is given. The symbol ✱ signals that an exercise is more challenging than usual.

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17. Let $A = \{3, 4, 5\}$ and $B = \{4, 5, 6\}$ and let R be the "less than" relation. That is,

$$\text{For all } (x, y) \in A \times B, \quad x \, R \, y \quad \Leftrightarrow \quad x < y.$$

State explicitly which ordered pairs are in R and R^{-1}.

18. Let $A = \{3, 4, 5\}$ and $B = \{4, 5, 6\}$ and let S be the "divides" relation. That is,

$$\text{For all } (x, y) \in A \times B, \quad x \, S \, y \quad \Leftrightarrow \quad x \mid y.$$

State explicitly which ordered pairs are in S and S^{-1}.

19. Let S be the set of all strings in a's and b's. Define a relation T on S as follows:

$$\text{For all } s, t \in S, \quad s \, T \, t \quad \Leftrightarrow \quad t = as$$

(that is, t is the concatenation of a with s).

a. Is $ab \, T \, aab$? **b.** Is $aab \, T \, ab$?
c. Is $ba \, T \, aba$? **d.** Is $aba \, T^{-1} \, ba$?
e. Is $abb \, T^{-1} \, bba$? f. Is $abba \, T^{-1} \, bba$?

20. Define a relation R from **R** to **R** as follows:

$$\text{For all } (x, y) \in \mathbf{R} \times \mathbf{R}, \quad x \, R \, y \quad \Leftrightarrow \quad y = \lfloor x \rfloor.$$

Draw the graphs of R and R^{-1} in the Cartesian plane.

21. **a.** Rewrite the definition of one-to-one function using the notation of the definition of a function as a relation.
b. Rewrite the definition of onto function using the notation of the definition of function as a relation.

22. **a.** Suppose a function $F: X \to Y$ is one-to-one but not onto. Is F^{-1} (the inverse relation for F) a function? Explain your answer.
b. Suppose a function $F: X \to Y$ is onto but not one-to-one. Is F^{-1} (the inverse relation for F) a function? Explain your answer.

Draw the directed graphs of the binary relations defined in 23–27 below.

23. Define a binary relation R on $A = \{0, 1, 2, 3\}$ by $R = \{(0, 0), (1, 2), (2, 2)\}$.

24. Define a binary relation S on $B = \{a, b, c, d\}$ by $S = \{(a, b), (a, c), (b, c), (d, d)\}$.

25. Let $A = \{2, 3, 4, 5, 6, 7, 8\}$ and define a binary relation R on A as follows:

$$\text{For all } x, y \in A, \quad x \, R \, y \quad \Leftrightarrow \quad x \mid y.$$

H **26.** Let $A = \{5, 6, 7, 8, 9, 10\}$ and define a binary relation S on A as follows:

$$\text{For all } x, y \in A, \quad x \, S \, y \quad \Leftrightarrow \quad 2 \mid (x - y).$$

27. Let $A = \{2, 3, 4, 5, 6, 7, 8\}$ and define a binary relation T on A as follows:

$$\text{For all } x, y \in A, \quad x \, T \, y \quad \Leftrightarrow \quad 3 \mid (x - y).$$

28. In Example 10.1.11 the result of the query SELECT Patient_ID#, Name FROM S WHERE Primary_Diagnosis = X is the projection onto the first two coordinates of the intersection of the set $A_1 \times A_2 \times A_3 \times \{X\}$ with the database.
a. Find the result of the query SELECT Patient_ID#, Name FROM S WHERE Primary_Diagnosis = pneumonia.
b. Find the result of the query SELECT Patient_ID#, Name FROM S WHERE Primary_Diagnosis = appendicitis.

Exercises 29–33 refer to unions and intersections of relations. Since binary relations are subsets of Cartesian products, their unions and intersections can be calculated as for any subsets. Given two relations R and S from A to B,

$$R \cup S = \{(x, y) \in A \times B \mid (x, y) \in R \text{ or } (x, y) \in S\}$$
$$R \cap S = \{(x, y) \in A \times B \mid (x, y) \in R \text{ and } (x, y) \in S\}.$$

29. Let $A = \{2, 4\}$ and $B = \{6, 8, 10\}$ and define binary relations R and S from A to B as follows:

$$\text{For all } (x, y) \in A \times B, \quad x \, R \, y \quad \Leftrightarrow \quad x \mid y.$$
$$\text{For all } (x, y) \in A \times B, \quad x \, S \, y \quad \Leftrightarrow \quad y - 4 = x.$$

State explicitly which ordered pairs are in $A \times B$, R, S, $R \cup S$, and $R \cap S$.

30. Let $A = \{-1, 1, 2, 4\}$ and $B = \{1, 2\}$ and define binary relations R and S from A to B as follows:

$$\text{For all } (x, y) \in A \times B, \quad x \, R \, y \quad \Leftrightarrow \quad |x| = |y|.$$
$$\text{For all } (x, y) \in A \times B, \quad x \, S \, y \quad \Leftrightarrow \quad x - y \text{ is even.}$$

State explicitly which ordered pairs are in $A \times B$, R, S, $R \cup S$, and $R \cap S$.

31. Define R and S from **R** to **R** as follows:

$$R = \{(x, y) \in \mathbf{R} \times \mathbf{R} \mid x < y\} \quad \text{and}$$
$$S = \{(x, y) \in \mathbf{R} \times \mathbf{R} \mid x = y\}.$$

That is, R is the "less than" relation and S is the "equals" relation from **R** to **R**. Graph R, S, $R \cup S$, and $R \cap S$ in the Cartesian plane.

32. Define binary relations R and S from **R** to **R** as follows:

$$R = \{(x, y) \in \mathbf{R} \times \mathbf{R} \mid x^2 + y^2 = 4\} \quad \text{and}$$
$$S = \{(x, y) \in \mathbf{R} \times \mathbf{R} \mid x = y\}.$$

Graph R, S, $R \cup S$, and $R \cap S$ in the Cartesian plane.

33. Define binary relations R and S from **R** to **R** as follows:

$$R = \{(x, y) \in \mathbf{R} \times \mathbf{R} \mid y = |x|\} \quad \text{and}$$
$$S = \{(x, y) \in \mathbf{R} \times \mathbf{R} \mid y = 1\}.$$

Graph R, S, $R \cup S$, and $R \cap S$ in the Cartesian plane.

By definition of R this means that

$$\text{For all } m, n \in \mathbf{Z}, \quad \text{if } 3 \mid (m - n) \text{ and } 3 \mid (n - p) \text{ then } 3 \mid (m - p).$$

Is this true? Suppose m, n, and p are particular but arbitrarily chosen integers such that $3 \mid (m - n)$ and $3 \mid (n - p)$. Must it follow that $3 \mid (m - p)$? By definition of "divides," since

$$3 \mid (m - n) \quad \text{and} \quad 3 \mid (n - p),$$

then

$$m - n = 3r \quad \text{for some integer } r,$$

and

$$n - p = 3s \quad \text{for some integer } s.$$

The crucial observation is that $(m - n) + (n - p) = m - p$. Add these two equations together to obtain

$$(m - n) + (n - p) = 3r + 3s,$$

which is equivalent to

$$m - p = 3(r + s).$$

Since r and s are integers, $r + s$ is an integer, and so this equation shows that

$$3 \mid (m - p).$$

It follows that R is transitive.

The reasoning above is formalized in the following proof.

Proof of Transitivity: Suppose m, n, and p are particular but arbitrarily chosen integers that satisfy the condition $m\ R\ n$ and $n\ R\ p$. [*We must show that* $m\ R\ p$.] By definition of R, since $m\ R\ n$ and $n\ R\ p$, then $3 \mid (m - n)$ and $3 \mid (n - p)$. By definition of "divides," this means that $m - n = 3r$ and $n - p = 3s$, for some integers r and s. Adding the two equations gives $(m - n) + (n - p) = 3r + 3s$, and simplifying gives that $m - p = 3(r + s)$. Since $r + s$ is an integer, this equation shows that $3 \mid (m - p)$. Hence, by definition of R, $m\ R\ p$ [*as was to be shown*]. ■

Exercise Set 10.2

In 1–8 a number of binary relations are defined on the set $A = \{0, 1, 2, 3\}$. For each relation:

a. Draw the directed graph.
b. Determine whether the relation is reflexive.
c. Determine whether the relation is symmetric.
d. Determine whether the relation is transitive.

Give a counterexample in each case in which the relation does not satisfy one of the properties.

1. $R_1 = \{(0, 0), (0, 1), (0, 3), (1, 1), (1, 0), (2, 3), (3, 3)\}$

2. $R_2 = \{(0, 0), (0, 1), (1, 1), (1, 2), (2, 2), (2, 3)\}$

3. $R_3 = \{(2, 3), (3, 2)\}$

4. $R_4 = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$

5. $R_5 = \{(0, 0), (0, 1), (0, 2), (1, 2)\}$

6. $R_6 = \{(0, 1), (0, 2)\}$

7. $R_7 = \{(0, 3), (2, 3)\}$

8. $R_8 = \{(0, 0), (1, 1)\}$

In 9–11, R, S, and T are binary relations defined on $A = \{0, 1, 2, 3\}$.

9. Let $R = \{(0, 1), (0, 2), (1, 1), (1, 3), (2, 2), (3, 0)\}$. Find R^t, the transitive closure of R.

10. Let $S = \{(0, 0), (0, 3), (1, 0), (1, 2), (2, 0), (3, 2)\}$. Find S^t, the transitive closure of S.

11. Let $T = \{(0, 2), (1, 0), (2, 3), (3, 1)\}$. Find T^t, the transitive closure of T.

In 12–36 determine whether the given binary relation is reflexive, symmetric, transitive, or none of these. Justify your answers.

12. R is the "greater than or equal to" relation on the set of real numbers: For all $x, y \in \mathbf{R}$, $x\ R\ y \Leftrightarrow x \geq y$.

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$x, y \in \mathbf{R}$

14. D is the
$x, y \in \mathbf{R}$

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13. C is the circle relation on the set of real numbers: For all $x, y \in \mathbf{R}, x\ C\ y \Leftrightarrow x^2 + y^2 = 1$.

14. D is the binary relation defined on $\mathbf{R}$ as follows: For all $x, y \in \mathbf{R}, x\ D\ y \Leftrightarrow xy \geq 0$.

15. E is the congruence modulo 2 relation on $\mathbf{Z}$: For all $m, n \in \mathbf{Z}, m\ E\ n \Leftrightarrow 2 \mid (m - n)$.

16. F is the congruence modulo 5 relation on $\mathbf{Z}$: For all $m, n \in \mathbf{Z}, m\ F\ n \Leftrightarrow 5 \mid (m - n)$.

17. O is the binary relation defined on $\mathbf{Z}$ as follows: For all $m, n \in \mathbf{Z}, m\ O\ n \Leftrightarrow m - n$ is odd.

18. D is the "divides" relation on $\mathbf{Z}^+$: For all positive integers m and n, $m\ D\ n \Leftrightarrow m \mid n$.

19. A is the "absolute value" relation on $\mathbf{R}$: For all real numbers x and y, $x\ A\ y \Leftrightarrow |x| = |y|$.

20. Recall that a prime number is an integer that is greater than 1 and has no positive integer divisors other than 1 and itself. (In particular, 1 is not prime.) A binary relation P is defined on $\mathbf{Z}$ as follows: For all $m, n \in \mathbf{Z}, m\ P\ n \Leftrightarrow \exists$ a prime number p such that $p \mid m$ and $p \mid n$.

21. Let S be the set of all strings of a's and b's. A binary relation L is defined on S as follows: For all strings $s, t \in S, s\ L\ t \Leftrightarrow l(s) < l(t)$ where l is the length function (that is, the number of characters in s is less than the number of characters in t).

22. Let B be the set of all strings of 0's and 1's. A binary relation G is defined on B as follows: For all $s, t \in B, s\ G\ t \Leftrightarrow$ the number of 0's in s is greater than the number of 0's in t.

23. Let $X = \{a, b, c\}$ and $\mathscr{P}(X)$ be the power set of X (the set of all subsets of X). A binary relation # is defined on $\mathscr{P}(X)$ as follows: For all $A, B \in \mathscr{P}(X), A \# B \Leftrightarrow$ the number of elements in A equals the number of elements in B.

24. Let $X = \{a, b, c\}$ and $\mathscr{P}(X)$ be the power set of X. A binary relation $\mathscr{R}$ is defined on $\mathscr{P}(X)$ as follows: For all $A, B \in \mathscr{P}(X), A\ \mathscr{R}\ B \Leftrightarrow N(A) < N(B)$ (that is, the number of elements in A is less than the number of elements in B).

25. Let $X = \{a, b, c\}$ and $\mathscr{P}(X)$ be the power set of X. A binary relation $\mathscr{N}$ is defined on $\mathscr{P}(X)$ as follows: For all $A, B \in \mathscr{P}(X), A\ \mathscr{N}\ B \Leftrightarrow N(A) \neq N(B)$ (that is, the number of elements in A is not equal to the number of elements in B).

26. Let A be a nonempty set and $\mathscr{P}(A)$ the power set of A. Define the "subset" relation $\mathscr{S}$ on $\mathscr{P}(A)$ as follows: For all $X, Y \in \mathscr{P}(A), X\ \mathscr{S}\ Y \Leftrightarrow X \subseteq Y$.

27. Let A be a nonempty set and $\mathscr{P}(A)$ the power set of A. Define the "not equal to" relation $\mathscr{R}$ on $\mathscr{P}(A)$ as follows: For all $X, Y \in \mathscr{P}(A), X\ \mathscr{R}\ Y \Leftrightarrow X \neq Y$.

28. Let A be a nonempty set and $\mathscr{P}(A)$ the power set of A. Define the "relative complement" relation $\mathscr{C}$ on $\mathscr{P}(A)$ as follows: For all $X, Y \in \mathscr{P}(A), X\ \mathscr{C}\ Y \Leftrightarrow Y = A - X$.

29. Let A be a set with at least two elements and $\mathscr{P}(A)$ the power set of A. Define a relation $\mathscr{R}$ on $\mathscr{P}(A)$ as follows: For all $X, Y \in \mathscr{P}(A), X\ \mathscr{R}\ Y \Leftrightarrow X \subseteq Y$ or $Y \subseteq X$.

30. Let A be the set of all English statements. A binary relation I is defined on A as follows: For all $p, q \in A$, $p\ I\ q \Leftrightarrow p \to q$ is true.

31. Let $A = \mathbf{R} \times \mathbf{R}$. A binary relation $\mathscr{R}$ is defined on A as follows: For all (x_1, y_1) and (x_2, y_2) in A, $(x_1, y_1)\ \mathscr{R}\ (x_2, y_2) \Leftrightarrow x_1 = x_2$.

32. Let $A = \mathbf{R} \times \mathbf{R}$. A binary relation $\mathscr{R}$ is defined on A as follows: For all (x_1, y_1) and (x_2, y_2) in A, $(x_1, y_1)\ \mathscr{R}\ (x_2, y_2) \Leftrightarrow y_1 = y_2$.

33. Let A be the "punctured plane"; that is, A is the set of all points in the Cartesian plane except the origin $(0, 0)$. A binary relation R is defined on A as follows: For all p_1 and p_2 in A, $p_1\ R\ p_2 \Leftrightarrow p_1$ and p_2 lie on the same half line emanating from the origin.

34. Let A be the set of people living in the world today. A binary relation R is defined on A as follows: For all $p, q \in A, p\ R\ q \Leftrightarrow p$ lives within 100 miles of q.

35. Let A be the set of all lines in the plane. A binary relation R is defined on A as follows: For all l_1 and l_2 in A, $l_1\ R\ l_2 \Leftrightarrow l_1$ is parallel to l_2. (Assume that a line is parallel to itself.)

36. Let A be the set of all lines in the plane. A binary relation R is defined on A as follows: For all l_1 and l_2 in A, $l_1\ R\ l_2 \Leftrightarrow l_1$ is perpendicular to l_2.

37. Let A be a set with eight elements.
   a. How many binary relations are there on A?
   b. How many binary relations on A are reflexive?
   c. How many binary relations on A are symmetric?
   d. How many binary relations on A are both reflexive and symmetric?

38. Write a computer algorithm to test whether a binary relation R defined on a finite set A is reflexive, where $A = \{a[1], a[2], \ldots, a[n]\}$.

39. Write a computer algorithm to test whether a binary relation R defined on a finite set A is symmetric, where $A = \{a[1], a[2], \ldots, a[n]\}$.

40. Write a computer algorithm to test whether a binary relation R defined on a finite set A is transitive, where $A = \{a[1], a[2], \ldots, a[n]\}$.

b. There is one equivalence class for each distinct rational number. Each equivalence class consists of all ordered pairs (a, b) that, if written as fractions a/b, would equal each other. The reason for this is that the condition for two rational numbers to be equal is the same as the condition for two ordered pairs to be related. For instance, the class of $(1, 2)$ is

$$[(1,2)] = \{(1,2), (-1,-2), (2,4), (-2,-4), (3,6), (-3,-6), \ldots\}$$

since $\frac{1}{2} = \frac{-1}{-2} = \frac{2}{4} = \frac{-2}{-4} = \frac{3}{6} = \frac{-3}{-6}$ and so forth. ■

It is possible to expand the result of Example 10.3.10 to define operations of addition and multiplication on the equivalence classes of R that satisfy all the same properties as the addition and multiplication of rational numbers. (See exercise 39.) It follows that the rational numbers can be defined as equivalence classes of ordered pairs of integers. Similarly (see exercise 40), it can be shown that all integers, negative and zero included, can be defined as equivalence classes of ordered pairs of positive integers. But in the late nineteenth century, F. L. G. Frege and Giuseppe Peano showed that the positive integers can be defined entirely in terms of sets. And just a little earlier, Richard Dedekind (1848–1916) showed that all real numbers can be defined as sets of rational numbers. All together, these results show that the real numbers can be defined using logic and set theory alone.

Exercise Set 10.3

1. Suppose that $S = \{a, b, c, d, e\}$ and R is a binary relation on S such that $a\,R\,b$, $b\,R\,c$, and $d\,R\,e$. List all of the following that must be true if R is (a) reflexive (but not symmetric or transitive), (b) symmetric (but not reflexive or transitive), (c) transitive (but not reflexive or symmetric), and (d) an equivalence relation.

$c\,R\,b \quad c\,R\,c \quad a\,R\,c \quad b\,R\,a \quad a\,R\,d \quad e\,R\,a \quad e\,R\,d \quad c\,R\,a$

2. Each of the following partitions of $\{0, 1, 2, 3, 4\}$ induces a relation R on $\{0, 1, 2, 3, 4\}$. In each case, find the ordered pairs in R.

a. $\{0, 2\}, \{1\}, \{3, 4\}$ b. $\{0\}, \{1, 3, 4\}, \{2\}$
c. $\{0\}, \{1, 2, 3, 4\}$

In 2–12, the relation R is an equivalence relation on the set A. Find the distinct equivalence classes of R.

3. $A = \{0, 1, 2, 3, 4\}$
$R = \{(0,0), (0,4), (1,1), (1,3), (2,2), (3,1), (3,3), (4,0), (4,4)\}$

4. $A = \{a, b, c, d\}$
$R = \{(a,a), (b,b), (b,d), (c,c), (d,b), (d,d)\}$

5. $A = \{1, 2, 3, 4, \ldots, 20\}$. R is defined on A as follows:

For all $x, y \in A$, $\quad x\,R\,y \;\Leftrightarrow\; 4 \mid (x - y)$.

6. $A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$. R is defined on A as follows:

For all $x, y \in A$, $\quad x\,R\,y \;\Leftrightarrow\; 3 \mid (x - y)$.

7. $A = \{(1,3), (2,4), (-4,-8), (3,9), (1,5), (3,6)\}$. R is defined on A as follows: For all $(a, b), (c, d) \in A$,

$$(a, b)\,R\,(c, d) \;\Leftrightarrow\; ad = bc.$$

8. $X = \{a, b, c\}$ and $A = \mathscr{P}(X)$. R is defined on A as follows:

For all sets U and V in $\mathscr{P}(X)$, $\quad U\,R\,V \;\Leftrightarrow\; N(U) = N(V)$.

(That is, the number of elements in U equals the number of elements in V.)

9. $X = \{-1, 0, 1\}$ and $A = \mathscr{P}(X)$. R is defined on $\mathscr{P}(X)$ as follows: For all sets S and T in $\mathscr{P}(X)$,

$S\,R\,T \;\Leftrightarrow\;$ the sum of the elements in S equals the sum of the elements in T.

10. A is the set of all strings of length 4 in a's and b's. R is defined on A as follows: For all strings s and t in A,

$s\,R\,t \;\Leftrightarrow\;$ the first two characters of s equal the first two characters of t.

11. A is the set of all strings of length 2 in 0's, 1's, and 2's. R is defined on A as follows: For all strings s and t in A,

$s\,R\,t \;\Leftrightarrow\;$ the sum of the characters in s equals the sum of the characters in t.

12. $A = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$. R is defined on A as follows:

For all $m, n \in \mathbf{Z}$, $\quad m\,R\,n \;\Leftrightarrow\; 3 \mid (m^2 - n^2)$.

13. Determine which of the following congruence relations are true and which are false.
a. $17 \equiv 2 \pmod 5$ b. $4 \equiv -5 \pmod 7$
c. $-2 \equiv -8 \pmod 3$ d. $-6 \equiv 22 \pmod 2$

14. **a.** Let R be the relation of congruence modulo 3. Which of the following equivalence classes are equal?

$$[7], [-4], [-6], [17], [4], [27], [19]$$

b. Let R be the relation of congruence modulo 7. Which of the following equivalence classes are equal?

$$[35], [3], [-7], [12], [0], [-2], [17]$$

15. **a.** Prove that for all integers m and n, $m \equiv n \pmod 3$ if, and only if, $m \bmod 3 = n \bmod 3$.
b. Prove that for all integers m and n and any positive integer d, $m \equiv n \pmod d$ if, and only if, $m \bmod d = n \bmod d$.

16. **a.** Give an example of two sets that are distinct but not disjoint.
b. Find sets A_1 and A_2 and elements x, y and z such that x and y are in A_1 and y and z are in A_2 but x and z are not both in either of the sets A_1 or A_2.

In 17–28, (1) prove that the relation is an equivalence relation, and (2) describe the distinct equivalence classes of each relation.

17. A is the set of all students at your college.
a. R is the relation defined on A as follows: For all x and y in A,

$x \; R \; y \Leftrightarrow$ x has the same major (or double major) as y.

(Assume "undeclared" is a major.)
b. S is the relation defined on A as follows: For all $x, y \in A$,

$x \; S \; y \Leftrightarrow$ x is the same age as y.

H **18.** E is the relation defined on $\mathbf{Z}$ as follows:

For all $m, n \in \mathbf{Z}$, $\quad m \; E \; n \Leftrightarrow 2 \mid (m - n)$.

19. F is the relation defined on $\mathbf{Z}$ as follows:

For all $m, n \in \mathbf{Z}$, $\quad m \; F \; n \Leftrightarrow 4 \mid (m - n)$.

20. Let A be the set of all statement forms in three variables p, q, and r. $\mathscr{R}$ is the relation defined on A as follows: For all P and Q in A,

$P \; \mathscr{R} \; Q \Leftrightarrow$ P and Q have the same truth table.

21. Let P be a set of parts shipped to a company from various suppliers. S is the relation defined on P as follows: For all $x, y \in P$,

$x \; S \; y \Leftrightarrow$ x has the same part number and is shipped from the same supplier as y.

22. A is the "absolute value" relation defined on $\mathbf{R}$ as follows:

For all $x, y \in \mathbf{R}$, $\quad x \; A \; y \Leftrightarrow |x| = |y|$.

23. I is the relation defined on $\mathbf{R}$ as follows:

For all $x, y \in \mathbf{R}$, $\quad x \; I \; y \Leftrightarrow x - y$ is an integer.

24. D is the relation defined on $\mathbf{Z}$ as follows:

For all $m, n \in \mathbf{Z}$, $\quad m \; D \; n \Leftrightarrow 3 \mid (m^2 - n^2)$.

25. Define P on the set $\mathbf{R} \times \mathbf{R}$ of ordered pairs of real numbers as follows: For all $(w, x), (y, z) \in \mathbf{R} \times \mathbf{R}$,

$$(w, x) \; P \; (y, z) \Leftrightarrow w = y.$$

26. Let A be the set of identifiers in a computer program. It is common for identifiers to be used for only a short part of the execution time of a program and not to be used again to execute other parts of the program. In such cases, arranging for identifiers to share memory locations makes efficient use of a computer's memory capacity. Define R on A as follows: For all identifiers x and y,

$x \; R \; y \Leftrightarrow$ the values of x and y are stored in the same memory location during execution of the program.

H **27.** Let A be the set of all straight lines in the Cartesian plane. Define a relation $\parallel$ on A as follows:

For all l_1 and l_2 in A, $\quad l_1 \parallel l_2 \Leftrightarrow l_1$ is parallel to l_2.

28. Let P be the set of all points in the Cartesian plane except the origin. R is the relation defined on P as follows: For all p_1 and p_2 in P,

$p_1 \; R \; p_2 \Leftrightarrow$ p_1 and p_2 lie on the same half-line emanating from the origin.

29. Let A be the set of points in the rectangle with x and y coordinates between 0 and 1. That is,

$$A = \{(x, y) \in \mathbf{R} \times \mathbf{R} \mid 0 \le x \le 1 \quad \text{and} \quad 0 \le y \le 1\}.$$

Define a relation R on A as follows: For all (x_1, y_1) and (x_2, y_2) in A,

$$(x_1, y_1) \; R \; (x_2, y_2) \Leftrightarrow$$
$$\begin{aligned} &(x_1, y_1) = (x_2, y_2); \quad \text{or} \\ &x_1 = 0 \text{ and } x_2 = 1 \text{ and } y_1 = y_2; \quad \text{or} \\ &x_1 = 1 \text{ and } x_2 = 0 \text{ and } y_1 = y_2; \quad \text{or} \\ &y_1 = 0 \text{ and } y_2 = 1 \text{ and } x_1 = x_2; \quad \text{or} \\ &y_1 = 1 \text{ and } y_2 = 0 \text{ and } x_1 = x_2. \end{aligned}$$

In other words, all points along the top edge of the rectangle are related to the points along the bottom edge directly beneath them, and all points directly opposite each other along the left and right edges are related to each other. The points in the interior of the rectangle are not related to anything other than themselves. Then R is an equivalence relation on A. Imagine gluing together all the points that are in the same equivalence class. Describe the resulting figure.