

## More on Solving Recurrence Relations Math 2300, Spring 2015

### Second Order Linear Recurrence Relations with Constant Coefficients

**Theorem:** Let  $p_n$  be any particular solution to the recurrence relation  $a_k = Aa_{k-1} + Ba_{k-2} + f(k)$ , ignoring initial conditions. Let  $q_n$  be the general solution to the homogeneous recurrence  $a_k = Aa_{k-1} + Ba_{k-2}$  again ignoring initial conditions. Then  $a_n = p_n + q_n$  is the general solution to the recurrence relation  $a_k = Aa_{k-1} + Ba_{k-2} + f(k)$ , where the initial conditions determine the constants in  $q_n$ .

#### Generating Functions:

A generating function is, roughly, a polynomial that goes on forever:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots$$

**Definition:** The generating function of a sequence  $a_0, a_1, a_2, a_3, \dots$  is the expression  $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \text{ generates } 1, 1, 1, \dots$$

$$\frac{1}{1-ax} = 1 + ax + a^2x^2 + a^3x^3 + \dots \text{ generates } 1, a, a^2, a^3, \dots$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots \text{ generates } 1, 2, 3, 4, \dots$$

(The above theorem and definitions adapted from *Discrete Mathematics with Graph Theory* by Goodaire and Parmenter.)

**Recursion Trees** are often used for divide and conquer algorithms.

**The Master Theorem** gives bounds on recurrences:

The Master Theorem applies to recurrences of the following form:

$$T(n) = aT(n/b) + f(n)$$

where  $a \geq 1$  and  $b > 1$  are constants and  $f(n)$  is an asymptotically positive function.

There are 3 cases:

1. If  $f(n) = O(n^{\log_b a - \epsilon})$  for some constant  $\epsilon > 0$ , then  $T(n) = \Theta(n^{\log_b a})$ .
2. If  $f(n) = \Theta(n^{\log_b a} \log^k n)$  with  $k \geq 0$ , then  $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$ .
3. If  $f(n) = \Omega(n^{\log_b a + \epsilon})$  with  $\epsilon > 0$ , and  $f(n)$  satisfies the regularity condition, then  $T(n) = \Theta(f(n))$ .  
Regularity condition:  $af(n/b) \leq cf(n)$  for some constant  $c < 1$  and all sufficiently large  $n$ .

For examples of applying the Master Theorem

[http://www.csanimated.com/animation.php?t=Master\\_theorem](http://www.csanimated.com/animation.php?t=Master_theorem)