Thus

$$S_{n,r} = S_{n-1,r-1} + rS_{n-1,r}$$

for all integers n and r with $1 < r < n$.

The initial conditions for the recurrence relation are

$$S_{n,1} = 1 \quad \text{and} \quad S_{n,n} = 1 \quad \text{for all integers } n \geq 1$$

because there is only one way to partition $\{x_1, x_2, \ldots, x_n\}$ into one subset, namely

$$\{x_1, x_2, \ldots, x_n\},$$

and only one way to partition $\{x_1, x_2, \ldots, x_n\}$ into n subsets, namely

$$\{x_1\}\{x_2\}, \ldots, \{x_n\}.$$

■

*Exercise Set 8.1**

Find the first four terms of each of the recursively defined sequences in 1–8.

1. $a_k = 2a_{k-1} + k$, for all integers $k \geq 2$
$a_1 = 1$

2. $b_k = b_{k-1} + 3k$, for all integers $k \geq 2$
$b_1 = 1$

3. $c_k = k(c_{k-1})^2$, for all integers $k \geq 1$
$c_0 = 1$

4. $d_k = k(d_{k-1})^2$, for all integers $k \geq 1$
$d_0 = 3$

5. $s_k = s_{k-1} + 2s_{k-2}$, for all integers $k \geq 2$
$s_0 = 1, s_1 = 1$

6. $t_k = t_{k-1} + 2t_{k-2}$, for all integers $k \geq 2$
$t_0 = -1, t_1 = 2$

7. $u_k = ku_{k-1} - u_{k-2}$, for all integers $k \geq 3$
$u_1 = 1, u_2 = 1$

8. $v_k = v_{k-1} + v_{k-2} + 1$, for all integers $k \geq 3$
$v_1 = 1, v_2 = 3$

9. Let $a_0, a_1, a_2, \ldots$ be defined by the formula $a_n = 3n + 1$, for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation $a_k = a_{k-1} + 3$, for all integers $k \geq 1$.

10. Let $b_0, b_1, b_2, \ldots$ be defined by the formula $b_n = 4^n$, for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation $b_k = 4b_{k-1}$, for all integers $k \geq 1$.

11. Show that the sequence $0, 1, 3, 7, \ldots, 2^n - 1, \ldots$, defined for $n \geq 0$, satisfies the recurrence relation

$$c_k = 2c_{k-1} + 1 \quad \text{for all integers } k \geq 1.$$

12. Show that the sequence $1, -1, \frac{1}{2}, \frac{-1}{3!}, \ldots, \frac{(-1)^n}{n!}, \ldots$, defined for $n \geq 0$, satisfies the recurrence relation

$$s_k = \frac{-s_{k-1}}{k} \quad \text{for all integers } k \geq 1.$$

13. Show that the sequence $2, 3, 4, 5, \ldots, 2 + n, \ldots$, defined for $n \geq 0$, satisfies the recurrence relation

$$t_k = 2t_{k-1} - t_{k-2} \quad \text{for all integers } k \geq 2.$$

14. Show that the sequence $0, 1, 5, 19, \ldots, 3^n - 2^n, \ldots$, defined for $n \geq 0$, satisfies the recurrence relation

$$d_k = 5d_{k-1} - 6d_{k-2} \quad \text{for all integers } k \geq 2.$$

15. Define a sequence $a_0, a_1, a_2, \ldots$ by the formula

$$a_n = (-2)^{\lfloor n/2 \rfloor} = \begin{cases} (-2)^{n/2} & \text{if } n \text{ is even} \\ (-2)^{(n-1)/2} & \text{if } n \text{ is odd} \end{cases}$$

for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation $a_k = -2a_{k-2}$, for all integers $k \geq 2$.

16. The sequence of Catalan numbers was defined in Exercise Set 6.6 by the formula $C_n = \frac{1}{n+1}\binom{2n}{n}$, for each integer $n \geq 1$. Show that this sequence satisfies the recurrence relation $C_k = \frac{4k-2}{k+1}C_{k-1}$, for all integers $k \geq 2$.

17. Use the recurrence relation and values for the Tower of Hanoi sequence $m_1, m_2, m_3, \ldots$ discussed in Example 8.1.5 to compute m_7 and m_8.

18. *Tower of Hanoi with Adjacency Requirement:* Suppose that in addition to the requirement that they never move a larger disk on top of a smaller one, the priests who move the disks

*For exercises with blue numbers or letters, solutions are given in Appendix B. The symbol ***H*** indicates that only a hint or a partial solution is given. The symbol ∗ signals that an exercise is more challenging than usual.

which is equivalent to

$$k = 0 \quad \text{by dividing both sides by 2.}$$

But this is false since k may be *any* nonnegative integer. Hence the sequence $c_0, c_1, c_2, \ldots$ does not satisfy the proposed formula. ■

Once you have found a proposed formula to be false, you should look back at your calculations to see where you made a mistake, correct it, and try again.

Exercise Set 8.2

1. The formula

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

is true for all integers $n \geq 1$. Use this fact to solve each of the following problems:

a. If k is an integer and $k \geq 2$, find a formula for the expression $1 + 2 + 3 + \cdots + (k-1)$.

b. If n is an integer and $n \geq 1$, find a formula for the expression $3 + 2 + 4 + 6 + 8 + \cdots + 2n$.

c. If n is an integer and $n \geq 1$, find a formula for the expression $3 + 3 \cdot 2 + 3 \cdot 3 + \cdots + 3 \cdot n + n$.

2. The formula

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

is true for all real numbers r except $r = 1$ and for all integers $n \geq 0$. Use this fact to solve each of the following problems:

a. If i is an integer and $i \geq 1$, find a formula for the expression $1 + 2 + 2^2 + \cdots + 2^{i-1}$.

b. If n is an integer and $n \geq 1$, find a formula for the expression $3^{n-1} + 3^{n-2} + \cdots + 3^2 + 3 + 1$.

c. If n is an integer and $n \geq 2$, find a formula for the expression $2^n + 2^{n-2} \cdot 3 + 2^{n-3} \cdot 3 + \cdots + 2^2 \cdot 3 + 2 \cdot 3 + 3$.

d. If n is an integer and $n \geq 1$, find a formula for the expression

$$2^n - 2^{n-1} + 2^{n-2} - 2^{n-3} + \cdots + (-1)^{n-1} \cdot 2 + (-1)^n.$$

In each of 3–15 a sequence is defined recursively. Use iteration to guess an explicit formula for the sequence. Use the formulas from Section 4.2 to simplify your answers whenever possible.

3. $a_k = k a_{k-1}$, for all integers $k \geq 1$
$a_0 = 1$

4. $b_k = \dfrac{b_{k-1}}{1 + b_{k-1}}$, for all integers $k \geq 1$
$b_0 = 1$

5. $c_k = 3c_{k-1} + 1$, for all integers $k \geq 2$
$c_1 = 1$

***H* 6.** $d_k = 2d_{k-1} + 3$, for all integers $k \geq 2$
$d_1 = 2$

7. $e_k = 4e_{k-1} + 5$, for all integers $k \geq 1$
$e_0 = 2$

8. $f_k = f_{k-1} + 2^k$, for all integers $k \geq 2$
$f_1 = 1$

***H* 9.** $g_k = \dfrac{g_{k-1}}{g_{k-1} + 2}$, for all integers $k \geq 2$
$g_1 = 1$

10. $h_k = 2^k - h_{k-1}$, for all integers $k \geq 1$
$h_0 = 1$

11. $p_k = p_{k-1} + 2 \cdot 3^k$
$p_1 = 2$

12. $s_k = s_{k-1} + 2k$, for all integers $k \geq 1$
$s_0 = 3$

13. $t_k = t_{k-1} + 3k + 1$, for all integers $k \geq 1$
$t_0 = 0$

✶14. $x_k = 3x_{k-1} + k$, for all integers $k \geq 2$
$x_1 = 1$

15. $y_k = y_{k-1} + k^2$, for all integers $k \geq 2$
$y_1 = 1$

16. Solve the recurrence relation obtained as the answer to exercise 18(c) of Section 8.1.

17. Solve the recurrence relation obtained as the answer to exercise 21(c) of Section 8.1.

18. Suppose d is a fixed constant and $a_0, a_1, a_2, \ldots$ is a sequence that satisfies the recurrence relation $a_k = a_{k-1} + d$, for all integers $k \geq 1$. Use mathematical induction to prove that $a_n = a_0 + nd$, for all integers $n \geq 0$.

19. A worker is promised a bonus if he can increase his productivity by 2 units a day every day for a period of 30 days. If on day 0 he produces 170 units, how many units must he produce on day 30 to qualify for the bonus?

20. A runner targets herself to improve her time on a certain course by 3 seconds a day. If on day 0 she runs the course in 3 minutes, how fast must she run it on day 14 to stay on target?

21. Suppose r is a fixed constant and $a_0, a_1, a_2 \ldots$ is a sequence that satisfies the recurrence relation $a_k = r a_{k-1}$, for all integers $k \geq 1$ and $a_0 = a$. Use mathematical induction to prove that $a_n = ar^n$, for all integers $n \geq 0$.

22. As shown in Example 8.1.8, if a bank pays interest at a rate of i compounded m times a year, then the amount of money P_k at the end of k time periods (where one time period $= 1/m$th of a year) satisfies the recurrence relation $P_k = [1 + (i/m)]P_{k-1}$ with initial condition $P_0 =$ the initial amount deposited. Find an explicit formula for P_n.

23. Suppose the population of a country increases at a steady rate of 3% per year. If the population is 50 million at a certain time, what will it be 25 years later?

24. A chain letter works as follows: One person sends a copy of the letter to five friends, each of whom sends a copy to five friends, each of whom sends a copy to five friends, and so forth. How many people will have received copies of the letter after the twentieth repetition of this process, assuming no person receives more than one copy?

25. A certain computer algorithm executes twice as many operations when it is run with an input of size k as when it is run with an input of size $k - 1$ (where k in an integer that is greater than 1). When the algorithm is run with an input of size 1, it executes seven operations. How many operations does it execute when it is run with an input of size 25?

26. A person saving for retirement makes an initial deposit of \$1,000 to a bank account earning interest at a rate of 3% per year compounded monthly, and each month she adds an additional \$200 to the account.
   a. For each nonnegative integer n, let A_n be the amount in the account at the end of n months. Find a recurrence relation relating A_k to A_{k-1}.
   H **b.** Use iteration to find an explicit formula for A_n.
   c. Use mathematical induction to prove the correctness of the formula you obtained in part (b).
   d. How much will the account be worth at the end of 20 years? At the end of 40 years?
   H **e.** In how many years will the account be worth \$10,000?

27. A person borrows \$3,000 on a bank credit card at a nominal rate of 18% per year, which is actually charged at a rate of 1.5% per month.
   H **a.** What is the annual percentage rate (APR) for the card? (See Example 8.1.8 for a definition of APR.)
   b. Assume that the person does not place any additional charges on the card and pays the bank \$150 each month to pay off the loan. Let B_n be the balance owed on the card after n months. Find an explicit formula for B_n.
   H **c.** How long will be required to pay off the debt?
   d. What is the total amount of money the person will have paid for the loan?

In 28–42 use mathematical induction to verify the correctness of the formula you obtained in the referenced exercise.

28. Exercise 3 29. Exercise 4 **30.** Exercise 5

31. Exercise 6 32. Exercise 7 33. Exercise 8

34. Exercise 9 ***H*** **35.** Exercise 10 36. Exercise 11

H **37.** Exercise 12 38. Exercise 13 **39.** Exercise 14

40. Exercise 15 41. Exercise 16 42. Exercise 17

In each of 43–49 a sequence is defined recursively. (a) Use iteration to guess an explicit formula for the sequence. (b) Use strong mathematical induction to verify that the formula of part (a) is correct.

43. $a_k = \dfrac{a_{k-1}}{2a_{k-1} - 1}$, for all integers $k \geq 1$
$a_0 = 2$

44. $b_k = \dfrac{2}{b_{k-1}}$, for all integers $k \geq 2$
$b_1 = 1$

45. $v_k = v_{\lfloor k/2 \rfloor} + v_{\lfloor (k+1)/2 \rfloor} + 2$, for all integers $k \geq 2$,
$v_1 = 1$.

H **46.** $s_k = 2s_{k-2}$, for all integers $k \geq 2$,
$s_0 = 1$, $s_1 = 2$.

47. $t_k = k - t_{k-1}$, for all integers $k \geq 1$,
$t_0 = 0$.

H **48.** $w_k = w_{k-2} + k$, for all integers $k \geq 3$,
$w_1 = 1$, $w_2 = 2$.

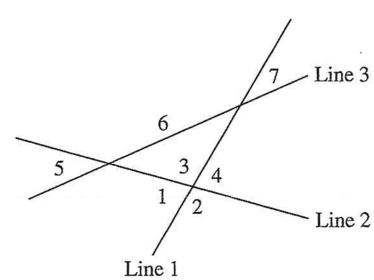
H **49.** $u_k = u_{k-2} \cdot u_{k-1}$, for all integers $k \geq 2$,
$u_0 = u_1 = 2$.

In 50 and 51 determine whether the given recursively defined sequence satisfies the explicit formula $a_n = (n - 1)^2$, for all integers $n \geq 1$.

50. $a_k = 2a_{k-1} + k - 1$, for all integers $k \geq 2$
$a_1 = 0$

51. $a_k = (a_{k-1} + 1)^2$, for all integers $k \geq 2$
$a_1 = 0$

52. A single line divides a plane into two regions. Two lines (by crossing) can divide a plane into four regions; three lines can divide it into seven regions (see the figure). Let P_n be the maximum number of regions into which n lines divide a plane, where n is a positive integer.



Exercise Set 8.3

1. Which of the following are second-order linear homogeneous recurrence relations with constant coefficients?

a. $a_k = 2a_{k-1} - 5a_{k-2}$
b. $b_k = kb_{k-1} + b_{k-2}$
c. $c_k = 3c_{k-1} \cdot c_{k-2}^2$
d. $d_k = 3d_{k-1} + d_{k-2}$
e. $r_k = r_{k-1} - r_{k-2} - 2$
f. $s_k = 10s_{k-2}$

2. Which of the following are second-order linear homogeneous recurrence relations with constant coefficients?

a. $a_k = (k-1)a_{k-1} + 2ka_{k-2}$
b. $b_k = -b_{k-1} + 7b_{k-2}$
c. $c_k = 3c_{k-1} + 1$
d. $d_k = 3d_{k-1}^2 + d_{k-2}$
e. $r_k = r_{k-1} - 6r_{k-3}$
f. $s_k = s_{k-1} + 10s_{k-2}$

3. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by the explicit formula

$$a_n = C \cdot 2^n + D \quad \text{for all integers } n \geq 0,$$

where C and D are real numbers.

a. Find C and D so that $a_0 = 1$ and $a_1 = 3$. What is a_2 in this case?
b. Find C and D so that $a_0 = 0$ and $a_1 = 2$. What is a_2 in this case?

4. Let $b_0, b_1, b_2, \ldots$ be the sequence defined by the explicit formula

$$b_n = C \cdot 3^n + D(-2)^n \quad \text{for all integers } n \geq 0,$$

where C and D are real numbers.

a. Find C and D so that $b_0 = 0$ and $b_1 = 5$. What is b_2 in this case?
b. Find C and D so that $b_0 = 3$ and $b_1 = 4$. What is b_2 in this case?

5. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by the explicit formula

$$a_n = C \cdot 2^n + D \quad \text{for all integers } n \geq 0,$$

where C and D are real numbers. Show that for any choice of C and D,

$$a_k = 3a_{k-1} - 2a_{k-2} \quad \text{for all integers } k \geq 2.$$

6. Let $b_0, b_1, b_2, \ldots$ be the sequence defined by the explicit formula

$$b_n = C \cdot 3^n + D(-2)^n \quad \text{for all integers } n \geq 0,$$

where C and D are real numbers. Show that for any choice of C and D,

$$b_k = b_{k-1} + 6b_{k-2} \quad \text{for all integers } k \geq 2.$$

7. Solve the system of equations in Example 8.3.4 to obtain

$$C = \frac{1+\sqrt{5}}{2\sqrt{5}} \quad \text{and} \quad D = \frac{-(1-\sqrt{5})}{2\sqrt{5}}.$$

In each of 8–10: (a) suppose a sequence of the form $1, t, t^2, t^3, \ldots, t^n, \ldots$, where $t \neq 0$, satisfies the given recurrence relation (but not necessarily the initial conditions), and find all possible values of t; (b) suppose a sequence satisfies the given initial conditions as well as the recurrence relation, and find an explicit formula for the sequence.

8. $a_k = 2a_{k-1} + 3a_{k-2}$, for all integers $k \geq 2$
$a_0 = 1, a_1 = 2$

9. $b_k = 7b_{k-1} - 10b_{k-2}$, for all integers $k \geq 2$
$b_0 = 2, b_1 = 2$

10. $c_k = c_{k-1} + 6c_{k-2}$, for all integers $k \geq 2$
$c_0 = 0, c_1 = 3$

In each of 11–15 suppose a sequence satisfies the given recurrence relation and initial conditions. Find an explicit formula for the sequence.

11. $d_k = 4d_{k-2}$, for all integers $k \geq 2$
$d_0 = 1, d_1 = -1$

12. $e_k = 9e_{k-2}$, for all integers $k \geq 2$
$e_0 = 0, e_1 = 2$

13. $r_k = 2r_{k-1} - r_{k-2}$, for all integers $k \geq 2$
$r_0 = 1, r_1 = 4$

14. $s_k = -4s_{k-1} - 4s_{k-2}$, for all integers $k \geq 2$
$s_0 = 0, s_1 = -1$

15. $t_k = 6t_{k-1} - 9t_{k-2}$, for all integers $k \geq 2$
$t_0 = 1, t_1 = 3$

H **16.** Find an explicit formula for the sequence of exercise 37 in Section 8.1.

17. Find an explicit formula for the sequence of exercise 39 in Section 8.1.

18. Suppose that the sequences $s_0, s_1, s_2, \ldots$ and $t_0, t_1, t_2, \ldots$ both satisfy the same second-order linear homogeneous recurrence relation with constant coefficients:

$$s_k = 5s_{k-1} - 4s_{k-2} \quad \text{for all integers } k \geq 2,$$
$$t_k = 5t_{k-1} - 4t_{k-2} \quad \text{for all integers } k \geq 2.$$

Show that the sequence $2s_0 + 3t_0, 2s_1 + 3t_1, 2s_2 + 3t_2, \ldots$ also satisfies the same relation. In other words, show that

$$2s_k + 3t_k = 5(2s_{k-1} + 3t_{k-1}) - 4(2s_{k-2} + 3t_{k-2})$$

for all integers $k \geq 2$. Do *not* use Lemma 8.3.2.

19. Show that if r, s, a_0, and a_1 are numbers with $r \neq s$, then there exist unique numbers C and D so that

$$C + D = a_0$$
$$Cr + Ds = a_1.$$

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5. Consider the set P of parenthesis structures defined in Example 8.4.4. Give derivations showing that each of the following is in P.
   a. ( ) (( )) b. (( ))(( ))

∗ 6. Determine whether either of the following parenthesis structures is in the set P defined in Example 8.4.4.
   a. ( )(( ) b. (( ) ( )))(( )

7. The set of arithmetic expressions over the real numbers can be defined recursively as follows:

   I. BASE: Each real number r is an arithmetic expression.

   II. RECURSION: If u and v are arithmetic expressions, then the following are also arithmetic expressions:
   a. $(+u)$ b. $(-u)$ c. $(u+v)$
   d. $(u-v)$ e. $(u \cdot v)$ f. $\left(\frac{u}{v}\right)$

   III. RESTRICTION: There are no arithmetic expressions over the real numbers other than those obtained from I and II.

   (Note that the *expression* $\left(\frac{u}{v}\right)$ is legal even though the value of v may be 0.) Give derivations showing that each of the following is an arithmetic expression.
   a. $((2 \cdot (0.3 - 4.2)) + (-7))$ b. $\left(\frac{(9 \cdot (6.1 + 2))}{((4 - 7) \cdot 6)}\right)$

8. Define a set S recursively as follows:

   I. BASE: $1 \in S$

   II. RECURSION: If $s \in S$, then
   a. $0s \in S$ b. $1s \in S$

   III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

   Use structural induction to prove that every string in S ends in a 1.

9. Define a set S recursively as follows:

   I. BASE: $a \in S$

   II. RECURSION: If $s \in S$, then,
   a. $sa \in S$ b. $sb \in S$

   III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

   Use structural induction to prove that every string in S begins with an a.

10. Define a set S recursively as follows:

   I. BASE: $\epsilon \in S$

   II. RECURSION: If $s \in S$, then
   a. $bs \in S$ b. $sb \in S$
   c. $saa \in S$ d. $aas \in S$

   III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

   Use structural induction to prove that every string in S contains an even number of a's.

11. Define a set S recursively as follows:

   I. BASE: $1 \in S,\ 2 \in S,\ 3 \in S,\ 4 \in S,\ 5 \in S,\ 6 \in S,\ 7 \in S, 8 \in S, 9 \in S$

   II. RECURSION: If $s \in S$ and $t \in S$, then
   a. $s0 \in S$ b. $st \in S$

   III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

   Use structural induction to prove that no string in S represents an integer with a leading zero.

H 12. Define a set S recursively as follows:

   I. BASE: $1 \in S, 3 \in S, 5 \in S, 7 \in S, 9 \in S$

   II. RECURSION: If $s \in S$ and $t \in S$ then
   a. $st \in S$ b. $2s \in S$ c. $4s \in S$
   d. $6s \in S$ e. $8s \in S$

   III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

   Use structural induction to prove that every string in S represents an odd integer.

H 13. Define a set S recursively as follows:

   I. BASE: $0 \in S, 5 \in S$

   II. RECURSION: If $s \in S$ and $t \in S$ then
   a. $s + t \in S$ b. $s - t \in S$

   III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

   Use structural induction to prove that every integer in S is divisible by 5.

14. Define a set S recursively as follows:

   I. BASE: $0 \in S$

   II. RECURSION: If $s \in S$, then
   a. $s + 3 \in S$ b. $s - 3 \in S$

   III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

   Use structural induction to prove that every integer in S is divisible by 3.

15. Give a recursive definition for the set of all strings of 0's and 1's that have the same number of 0's as 1's.

16. Give a recursive definition for the set of all strings of 0's and 1's for which all the 0's precede all the 1's.

17. Give a recursive definition for the set of all strings of a's and b's that contain an odd number of a's.

18. Give a recursive definition for the set of all strings of a's and b's that contain exactly one a.

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