

First-Order Logic

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<http://aima.eecs.berkeley.edu/2nd-ed/slides-ppt/>

Outline

- Why FOL?
- Syntax and semantics of FOL
- Using FOL
- Wumpus world in FOL
- Knowledge engineering in FOL

Pros and cons of propositional logic

- ⊙ Propositional logic is **declarative**
- ⊙ Propositional logic allows partial/disjunctive/negated information
 - (unlike most data structures and databases)
- ⊙ Propositional logic is **compositional**:
 - meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$
- ⊙ Meaning in propositional logic is **context-independent**
 - (unlike natural language, where meaning depends on context)
- ⊙ Propositional logic has very limited expressive power
 - (unlike natural language)
 - E.g., cannot say "pits cause breezes in adjacent squares"
 - except by writing one sentence for each square

First-order logic

- Whereas propositional logic assumes the world contains **facts**,
- first-order logic (like natural language) assumes the world contains
 - **Objects**: people, houses, numbers, colors, baseball games, wars, ... (squares, pits, wumpuses)
 - **Relations**: red, round, prime, brother of, bigger than, part of, comes between, ... (is breezy, is adjacent to, shoots)
 - **Functions**: father of, best friend, one more than, plus, ...

Propositional Logic vs FOL

$B_{23} \rightarrow (P_{32} \vee P_{23} \vee P_{34} \vee P_{43}) \dots$

"Internal squares adjacent to pits are breezy":

All $X Y (B(X,Y) \wedge (X > 1) \wedge (Y > 1) \wedge (Y < 4) \wedge (X < 4))$

\leftrightarrow

$(P(X-1,Y) \vee P(X,Y-1) \vee P(X+1,Y) \vee P(X,Y+1))$

Syntax of FOL: Basic elements

- Constants KingJohn, 2, Pitt,...
- Predicates Brother, >,...
- Functions Sqrt, LeftLegOf,...
- Variables x, y, a, b, \dots
- Connectives $\neg, \Rightarrow, \wedge, \vee, \Leftrightarrow$
- Equality =
- Quantifiers \forall, \exists

[BNF Grammar on p 293](#)

Sentence \rightarrow AtomicSentence |
 (Sentence Connective Sentence) |
 Quantifier Variable, .. Sentence |
 ~Sentence
 AtomicSentence \rightarrow Predicate(Term,...) | Term = Term
 Term \rightarrow Function(Term,...) |
 Constant |
 Variable
 Connective \rightarrow \rightarrow | \wedge | \vee | \leftrightarrow
 Quantifier \rightarrow all, exists
 Constant \rightarrow john, 1, ...
 Variable \rightarrow A, B, C, X
 Predicate \rightarrow breezy, sunny, red
 Function \rightarrow fatherOf, plus

Knowledge engineering involves deciding what types of things
Should be constants, predicates, and functions for your problem

Detour: Some British History

- Richard the Lionheart
 - Richard I (8 September 1157 - 6 April 1199) was King of England from 6 July 1189 until his death in 1199.
 - Rebelled unsuccessfully against father Henry II
 - Spoke very little English and mostly lived in Aquitaine
 - Was a central Christian commander during the Third Crusade
 - In his absence, brother John tries to seize throne, Richard forgives him

Atomic sentences

Atomic sentence = $predicate(term_1, \dots, term_n)$
| $term_1 = term_2$

Term = $function(term_1, \dots, term_n)$
| $constant$ | $variable$

- Brother(John, Richard)
- Married(Father(Richard), Mother(John))

Complex sentences

- Complex sentences are made from atomic sentences using connectives

$\neg S, S_1 \wedge S_2, S_1 \vee S_2, S_1 \Rightarrow S_2, S_1 \Leftrightarrow S_2,$

E.g. Sibling(KingJohn, Richard) \Rightarrow
Sibling(Richard, KingJohn)

$>(1,2) \vee \leq(1,2)$
 $>(1,2) \wedge \neg >(1,2)$

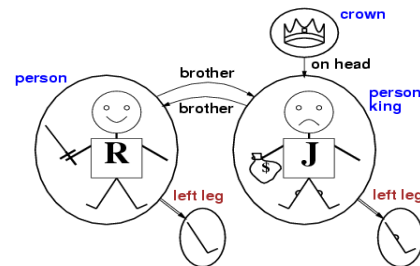
Truth in first-order logic

- Sentences are true with respect to a model and an interpretation
- Model contains objects (domain elements) and relations among them
- Interpretation specifies referents for
 - constant symbols \rightarrow objects
 - predicate symbols \rightarrow relations
 - function symbols \rightarrow functional relation

Interpretation: assignment of elements from the world to elements of the language

- An atomic sentence $predicate(term_1, \dots, term_n)$ is true iff the objects referred to by $term_1, \dots, term_n$ are in the relation referred to by $predicate$

Models for FOL: Example



Quantifiers

- All $\forall x p(x)$ means that p holds for all elements in the domain
- Exists $\exists x p(x)$ means that p holds for at least one element of the domain

Universal quantification

- $\forall \langle \text{variables} \rangle \langle \text{sentence} \rangle$

Everyone at CSU is smart:
 $\forall x \text{At}(x, \text{CSU}) \Rightarrow \text{Smart}(x)$

- $\forall x P$ is true in a model m iff P is true with x being each possible object in the model
- Roughly speaking, equivalent to the **conjunction** of **instantiations** of P
 $\text{At}(\text{KingJohn}, \text{CSU}) \Rightarrow \text{Smart}(\text{KingJohn})$
 $\wedge \text{At}(\text{Richard}, \text{CSU}) \Rightarrow \text{Smart}(\text{Richard})$
 $\wedge \text{At}(\text{CSU}, \text{CSU}) \Rightarrow \text{Smart}(\text{CSU})$
 $\wedge \dots$

A common mistake to avoid

- Typically, \Rightarrow is the main connective with \forall
- Common mistake: using \wedge as the main connective with \forall :

$\forall x \text{At}(x, \text{CSU}) \wedge \text{Smart}(x)$

means “Everyone is at CSU and everyone is smart”

Existential quantification

- $\exists \langle \text{variables} \rangle \langle \text{sentence} \rangle$

- Someone at CSU is smart:
 $\exists x \text{At}(x, \text{CSU}) \wedge \text{Smart}(x)$

- $\exists x P$ is true in a model m iff P is true with x being some possible object in the model

- Roughly speaking, equivalent to the **disjunction** of **instantiations** of P
 $\text{At}(\text{KingJohn}, \text{CSU}) \wedge \text{Smart}(\text{KingJohn})$
 $\vee \text{At}(\text{Richard}, \text{CSU}) \wedge \text{Smart}(\text{Richard})$
 $\vee \text{At}(\text{CSU}, \text{CSU}) \wedge \text{Smart}(\text{CSU})$
 $\vee \dots$

Another common mistake to avoid

- Typically, \wedge is the main connective with \exists
- Common mistake: using \Rightarrow as the main connective with \exists :

$\exists x \text{At}(x, \text{CSU}) \Rightarrow \text{Smart}(x)$

is true if there is anyone who is not at CSU!

Transform to:

$\exists x \sim(\text{At}(x, \text{CSU})) \vee \text{Smart}(x)$

Examples

- Everyone likes chocolate
- Someone likes chocolate
- Everyone likes chocolate unless they are allergic to it

Examples

- Everyone likes chocolate
 - $\forall X \text{ person}(X) \rightarrow \text{likes}(X, \text{chocolate})$
- Someone likes chocolate
 - $\exists X \text{ person}(X) \wedge \text{likes}(X, \text{chocolate})$
- Everyone likes chocolate unless they are allergic to it
 - $\forall X (\text{person}(X) \wedge \neg \text{allergic}(X, \text{chocolate})) \rightarrow \text{likes}(X, \text{chocolate})$

Properties of quantifiers

- $\forall x \forall y$ is the same as $\forall y \forall x$
- $\exists x \exists y$ is the same as $\exists y \exists x$
- $\exists x \forall y$ is **not** the same as $\forall y \exists x$
 - “There is a person who loves everyone in the world”
- $\forall y \exists x \text{ Loves}(x,y)$
 - “Everyone in the world is loved by at least one person”

Nesting of Variables

*Put quantifiers in front of likes(P,F)
Assume the domain of discourse of P is the set of people
Assume the domain of discourse of F is the set of foods*

1. Everyone likes some kind of food
2. There is a kind of food that everyone likes
3. Someone likes all kinds of food
4. Every food has someone who likes it

Answers

(DOD of P is people and F is food)

- Everyone likes some kind of food
 $\forall P, \exists F \text{ likes}(P,F)$
- There is a kind of food that everyone likes
 $\exists F, \forall P \text{ likes}(P,F)$
- Someone likes all kinds of food
 $\exists P, \forall F \text{ likes}(P,F)$
- Every food has someone who likes it
 $\forall F, \exists P \text{ likes}(P,F)$

Answers, without Domain of Discourse Assumptions

- Everyone likes some kind of food
 $\forall P \text{ person}(P) \rightarrow \exists F \text{ food}(F) \text{ and likes}(P,F)$
- There is a kind of food that everyone likes
 $\exists F \text{ food}(F) \text{ and } (\forall P \text{ person}(P) \rightarrow \text{likes}(P,F))$
- Someone likes all kinds of food
 $\exists P \text{ person}(P) \text{ and } (\forall F \text{ food}(F) \rightarrow \text{likes}(P,F))$
- Every food has someone who likes it
 $\forall F \text{ food}(F) \rightarrow \exists P \text{ person}(P) \text{ and likes}(P,F)$

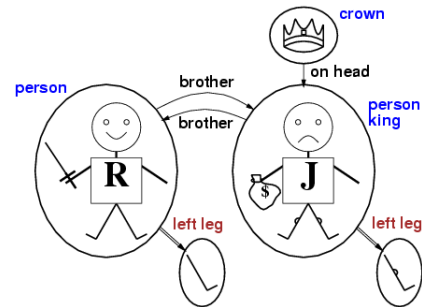
Quantification and Negation

- $\neg \exists x p(x)$ **equiv** $\forall x \neg p(x)$
 - $\neg \exists x \text{ likes}(x, \text{parsnips})$
 - $\forall x \neg \text{likes}(x, \text{parsnips})$
- $\neg \forall x p(x)$ **equiv** $\exists x \neg p(x)$
 - $\neg \forall x \text{ likes}(x, \text{parsnips})$
 - $\exists x \neg \text{likes}(x, \text{parsnips})$
- **Quantifier duality:** each can be expressed using the other
- $\forall x \text{ Likes}(x, \text{IceCream}) \rightarrow \neg \exists x \neg \text{Likes}(x, \text{IceCream})$
- $\exists x \text{ Likes}(x, \text{Broccoli}) \rightarrow \neg \forall x \neg \text{Likes}(x, \text{Broccoli})$

Equality

- $term_1 = term_2$ is true under a given interpretation if and only if $term_1$ and $term_2$ refer to the same object
- E.g., definition of *Sibling* in terms of *Parent*:

$$\forall x,y \text{ Sibling}(x,y) \Leftrightarrow [\neg(x = y) \wedge \exists m,f \neg (m = f) \wedge \text{Parent}(m,x) \wedge \text{Parent}(f,x) \wedge \text{Parent}(m,y) \wedge \text{Parent}(f,y)]$$



- Predicate of brotherhood:
 $\neg \langle R, J \rangle, \langle J, R \rangle$
- Predicate of being on: $\langle C, J \rangle$
- Predicate of being a person:
 $\neg \langle J, R \rangle$
- Predicate of being the king: $\langle J \rangle$
- Predicate of being a crown: $\langle C \rangle$
- Function for left legs: $\langle \langle J, JLL \rangle, \langle R, RLL \rangle \rangle$

Interpretation

- Specifies which objects, functions, and predicates are referred to by which constant symbols, function symbols, and predicate symbols.
- Under the intended interpretation:
 - “richard!” refers to R; “john!” refers to J; “crown” refers to the crown.
 - “onHead”, “brother”, “person”, “king”, “crown”, “leftLeg”, “strong”

Lots of other possible interpretations

- 5 objects, so just for constants “richard” and “john” there are 25 possibilities
- Note that the legs don’t have their own names!
- “john!” and “johnLackland” may be assigned the same object, J
- Also possible: “crown” and “john!” refer to C (just not the intended interpretation)

Why isn’t the “intended interpretation” enough?

- Vague notion. What is intended may be ambiguous (and often is, for non-toy domains)
- Logically possible: $\text{square}(x) \wedge \text{round}(x)$. Your KB has to include knowledge that rules this out.