First-Order Logic

Dr. Melanie Martin

CS 4480

November 5, 2010

Based on slides from

http://aima.eecs.berkeley.edu/2nd-ed/slides-ppt/

Detour: Some British History

- Richard the Lionheart
  - **Richard I** (8 September 1157 - 6 April 1199) was King of England from 6 July 1189 until his death in 1199.
  - Rebelled unsuccessfully against father Henry II
  - Spoke very little English and mostly lived in Aquitaine
  - Was a central Christian commander during the Third Crusade
  - In his absence, brother John tries to seize throne, Richard forgives him

Syntax of FOL: Basic elements

• Constants	KingJohn, 2, Pitt,...
• Predicates	Brother, >,...
• Functions	Sqrt, LeftLegOf,...
• Variables	x, y, a, b,...
• Connectives	$\neg$, $\Rightarrow$, $\wedge$, $\vee$, $\Leftrightarrow$
• Equality	=
• Quantifiers	$\forall$, $\exists$

BNF Grammar on p 247

Sentence → AtomicSentence |
(Sentence Connective Sentence) |
Quantifier Variable, .. Sentence |
~Sentence

AtomicSentence → Predicate(Term,…) | Term = Term

Term → Function(Term,…) |
Constant |
Variable

Connective → → | ^ | v | ←→

Quantifier → all, exists

Constant → john, 1, …

Variable → A, B, C, X

Predicate → breezy, sunny, red

Function → fatherOf, plus

Knowledge engineering involves deciding what types of things
Should be constants, predicates, and functions for your problem

Propositional Logic vs FOL

$B_{23} \rightarrow (P_{32} \vee P_{23} \vee P_{34} \vee P_{43})$ …

"Internal squares adjacent to pits are breezy":

All X Y (B(X,Y) ^ (X > 1) ^ (Y > 1) ^ (Y < 4) ^ (X < 4)) ←→

(P(X-1,Y) v P(X,Y-1) v P(X+1,Y) v (X,Y+1))

FOL (FOPC) Worlds

- Rather than just T,F, now worlds contain:
- Objects: the gold, the wumpus, …
  "the domain"
- Predicates: holding, breezy
- Functions: sonOf

Ontological commitment

Truth in first-order logic

- Sentences are true with respect to a model and an interpretation
- Model contains objects (domain elements) and relations among them
- Interpretation specifies referents for
  - constant symbols → objects
  - predicate symbols → relations
  - function symbols → functional relation

Interpretation: assignment of elements from the world to elements of the language

- An atomic sentence *predicate*($term_1,\ldots,term_n$) is true iff the objects referred to by $term_1,\ldots,term_n$ are in the relation referred to by *predicate*

Quantifiers

- All X p(X) means that p holds for all elements in the domain
- Exists X p(X) means that p holds for at least one element of the domain

Universal quantification

- ∀<*variables*> <*sentence*>

Everyone at CSU is smart:
∀x At(x,CSU) ⇒ Smart(x)

- ∀x *P* is true in a model *m* iff *P* is true with *x* being each possible object in the model

- Roughly speaking, equivalent to the conjunction of instantiations of *P*

  At(KingJohn,CSU) ⇒ Smart(KingJohn)
  ∧ At(Richard,CSU) ⇒ Smart(Richard)
  ∧ *At(CSU,CSU) ⇒ Smart(CSU)*
  ∧ ...

A common mistake to avoid

- Typically, $\Rightarrow$ is the main connective with $\forall$
- Common mistake: using $\wedge$ as the main connective with $\forall$:

  $\forall x \; At(x,CSU) \wedge Smart(x)$

  means "Everyone is at CSU and everyone is smart"

Existential quantification

- ∃<*variables*> <*sentence*>

- Someone at CSU is smart:
- $\exists x$ At(x,CSU) $\wedge$ Smart(x)

- $\exists x\ P$ is true in a model m iff P is true with x being some possible object in the model

- Roughly speaking, equivalent to the disjunction of instantiations of P

  At(KingJohn,CSU) $\wedge$ Smart(KingJohn)
  $\vee$ At(Richard,CSU) $\wedge$ Smart(Richard)
  $\vee$ At(CSU,CSU) $\wedge$ Smart(CSU)
  $\vee$ ...

Another common mistake to avoid

- Typically, $\wedge$ is the main connective with $\exists$
- Common mistake: using $\Rightarrow$ as the main connective with $\exists$:

$$\exists x \ \mathrm{At}(x,\mathrm{CSU}) \Rightarrow \mathrm{Smart}(x)$$

is true if there is anyone who is not at CSU!

Transform to:

$$\exists x \ \sim(\mathrm{At}(x,\mathrm{CSU})) \vee \mathrm{Smart}(x)$$

Examples

- Everyone likes chocolate
- Someone likes chocolate
- Everyone likes chocolate unless they are allergic to it

Examples

- Everyone likes chocolate
  - $\forall X$ person(X) → likes(X, chocolate)
- Someone likes chocolate
  - $\exists X$ person(X) ^ likes(X, chocolate)
- Everyone likes chocolate unless they are allergic to it
  - $\forall X$ (person(X) ^ ¬allergic (X, chocolate)) → likes(X, chocolate)

Properties of quantifiers

- $\forall x\ \forall y$ is the same as $\forall y\ \forall x$
- $\exists x\ \exists y$ is the same as $\exists y\ \exists x$

- $\exists x\ \forall y$ is not the same as $\forall y\ \exists x$
- $\exists x\ \forall y\ Loves(x,y)$
  - "There is a person who loves everyone in the world"
- $\forall y\ \exists x\ Loves(x,y)$
  - "Everyone in the world is loved by at least one person"

Nesting of Variables

Put quantifiers in front of likes(P,F)
Assume the domain of discourse of P is the set of people
Assume the domain of discourse of F is the set of foods

1. Everyone likes some kind of food
2. There is a kind of food that everyone likes
3. Someone likes all kinds of food
4. Every food has someone who likes it

Answers
(DOD of P is people and F is food)

Everyone likes some kind of food

$\forall$ P, $\exists$ F likes(P,F)

There is a kind of food that everyone likes

$\exists$ F, $\forall$ P likes(P,F)

Someone likes all kinds of food

$\exists$ P, $\forall$ F likes(P,F)

Every food has someone who likes it

$\forall$ F, $\exists$ P likes(P,F)

Answers, without Domain of Discourse Assumptions

Everyone likes some kind of food

$\forall$ P person(P) → $\exists$ F food(F) and likes(P,F)

There is a kind of food that everyone likes

$\exists$ F food(F) and ($\forall$ P person(P) → likes(P,F))

Someone likes all kinds of food

$\exists$ P person(P) and ($\forall$ F food(F) → likes(P,F))

Every food has someone who likes it

$\forall$ F food (F) → $\exists$ P person(P) and likes(P,F)

Quantification and Negation

- $\neg \exists x\ p(x)$ equiv $\forall x\ \neg p(x)$
  - $\neg \exists x$ likes(x, parsnips)
  - $\forall x\ \neg$likes(x, parsnips)
- $\neg \forall x\ p(x)$ equiv $\exists x\ \neg p(x)$
  - $\neg \forall x$ likes(x, parsnips)
  - $\exists x\ \neg$likes(x, parsnips)

- Quantifier duality: each can be expressed using the other
- $\forall x$ Likes(x,IceCream) $\neg \exists x\ \neg$Likes(x,IceCream)
- $\exists x$ Likes(x,Broccoli) $\neg \forall x\ \neg$Likes(x,Broccoli)

Equality

- $term_1 = term_2$ is true under a given interpretation if and only if $term_1$ and $term_2$ refer to the same object

- E.g., definition of *Sibling* in terms of *Parent*:
  $\forall x,y\ Sibling(x,y) \Leftrightarrow [\neg(x = y) \wedge \exists m,f\ \neg\ (m = f) \wedge Parent(m,x) \wedge Parent(f,x) \wedge Parent(m,y) \wedge Parent(f,y)]$