
Day One: Wed, Aug 23, 2017

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* **Take Roll**
+ Determine which enrolled students are present
+ Warn people that if they want to keep the class they should show up
+ Determine which students on the waiting list are present wanting to add
+ Determine **who else** there is present, whether enrolled, wanting to add, or whatever

ASSUMING NO ONE DROPS, NO ONE CAN BE ADDED. THE CLASS IS FULL.

+ Deal with permission numbers
- Maybe agree to **e-mail permission numbers** at some time
- Maybe announce permissions will be given out on paper on some particular day
- Maybe inform people they need to come back to see if they are added to the class
- Maybe announce that some will be allowed in and some not
- Maybe announce what day the decision will be made.

* Tell students that they are required to carefully read the entire course description on line (or request a printed full copy if not able to read on line)

* **Go over items in course description**
+ Name
+ Office & Hours
+ E-mail
+ Homepage + Class web site

* Go over Course Objectives - what do folks think those objectives mean?

* **Text**
+ Problems getting one?
+ How many will be using Ed #4? How many using Ed #3?
+ Do students know about the book web site - go and check out the errata and review materials with the class

* **Course components**
+ HW (25%)
+ 2 or more quizzes (50%)
+ comprehensive final exam (25%)

Exception: You fail the course if you get a failing average on tests. Otherwise I use the weights above.

HOMEWORK

Look at the Assignments with Class -- First Assignment is TODAY

Make sure to be familiar with homework submission rules in the course description -- notice strict late policy -- anybody here not able to access the course description, ask me for a paper copy.

Schedule - look at the schedule with the class and go over this week's assignments -- There is reading

and homework. The first HW is due next Wednesday (Aug 30).

Lecture

Chapter One - Chapter One is about language we use to make it easier to talk about mathematical concepts, and to make it easier to find solutions to math problems.

Section 1.1 - Variables
Section 1.2 - The Language of Sets
Section 1.3 - The Language of Relations and Functions

Section 1.1 - Variables are names we use for numbers or other things.

Example: (x)
We can use a variable as a names of a specific numbers that we don't know.
Example: x = 23981 times 53204

When we want to state a fact that is true about a lot of numbers, we can use a variable as a name that represents any one of those numbers.
Example: For all real numbers y, 2(y + 1) - y = y + 2

If we are wondering if a number with a certain property exists, we can use a variable to represent the number, to help us find the number, or to prove it does NOT exist.

Example: Can 6 be the sum of a number and its square?
With variable: $x^2 + x = 6$. Using the quadratic formula, we find that 2 and -3 work.

It's handy to use variables as names for numbers if we write some mathematical information that is long. We can use a variable name to refer to something over and over again in different parts of what we write. The reader understands what we mean, because we use the variable name consistently.

People often use variables to make mathematical statements
Universal: All real numbers y have the property that 2(y + 1) - y = y + 2
Conditional: If x=2 or x=-3 then x^2 + x = 6
Existential: There are two numbers x,y such that x^2 = 25, and y^2 = 25.

Day Two: Fri, Aug 25, 2017

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* Take Roll
* Take care of remaining problems with adding the class
* Today's HW assignment is problems for section 1.2 on page 13: 2,4,7,9,12; Due on Friday, September 1.
* Wednesday's assignment is some section 1.1 problems on page 5, 2,4,7,11,13; due Wednesday, Aug 31.

* The goal today is to come close to finishing discussion of chapter one.

Ask students for questions about the assigned HW.

Work problems with the class:

1.1.3. Fill in the blanks using a variable or variables to rewrite the given statement:

Given any two real numbers, there is a real number in between.

a. Given any two real numbers a and b, there is a real number c such that c is ___________ .

b. For any two ________, _______ such that a < c < b.

1.1.10. Every nonzero real number has a reciprocal.

a. All non-zero real numbers _______________

b. For all nonzero real numbers r, there is __________ for r.

c. For all nonzero real numbers r, there is a real number s such that _________.

Section 1.2 - The Language of sets

Sets are basically just collections of things - kind of like a basket, containing some things, or maybe empty.

The things that are in the set are called elements of the set.
If S is a set and x is an element of S, we use the notation x ∊ S to denote that.

We can indicate the elements of a set by using "roster" notation with or without ellipsis:

S={a, t, r} ;
T = {1, 2, ..., 14};
V = {31.2, 41.2, 51.2, ... }

When you use ellipsis you need to be sure that the people who read your writings will understand what you mean by the ellipsis. That's why something is written in the beginning to establish the pattern.

The idea of a set does not include any notion of the order of the elements.
S={1,2,3}; T={2,1,3}; and U={2,3,1} are all the same set, because the elements are the same.

Also, an element is either in the set or not. An element can't be in a set twice.
For example {1,2,2} = {1,2}

Familiar sets R, Z, Q, R+, Z-, etc

R and Z illustrate the difference between the idea of continuous and discrete mathematics

Tell students to work as many HW problems as possible between now and Monday, and come in with their questions about how to work them.

Show class some example of "**set builder**" **notation** - say using inequality conditions

Write in words how to read each of the following out loud.
1.2.2.a: {x ∈ R^+ | 0 < x < 1}
1.2.2.c: {n ∈ Z | n is a factor of 6}

There's roster notation to specify a set - like **S = {-1, 0, 1, 2, 3}.**
You can also use "SET BUILDER" notation to specify a set:

For example the set S above is also described as

S = { n ∈ Z | -1 ≤ n ≤ 3 }

It's the same set - You read the notation this way:

"The set of all n in Z (integers) such that n is between -1 and 3 (inclusive).

The **set builder notation** is generally **more expressive** than the roster notation. In other words, it's often easier to say what you mean using the set builder notation. For example:

C1 = { m ∈ Z | m = pq, where p and q are primes}

Imagine trying to use roster notation to specify the set C1.

Example problems:
Use the set-roster notation to indicate the elements in each of the following sets.
1.2.7.a: S = {n ∈ Z | n = $(-1)^k$, for some integer k}
1.2.7.c: U = {r ∈ Z | 2 ≤ r ≤ -2}

Day Three: Mon, Aug 28, 2017
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* Take Roll

Example problem:
Use the set-roster notation to indicate the elements in the following set. 1.2.7.d: V = {s ∈ Z | s >2 or s < 3}

Notions of **subset, proper subset, equal sets,** difference between being a **subset** and being an **element.**

Subset: if A and B are sets, then
A ⊆ B means that **every element of A is also an element of B**

Example:
{2, 3, 1} ⊆ {-20, 2.3, 1, 3, 2, 4}
Example: Z ⊆ R
Example: {1,2} ⊆ {1,2}

Proper Subset: if A is a subset of B, and if **there is at least one element of B that is NOT in A,** then A is a proper subset of B.

Example of Proper Subset Relation: {2, 3, 1} ⊆ {-20, 2.3, 1, 3, 2, 4}
Example of Proper Subset Relation: Z ⊆ R

Equality of Sets: If A and B are sets such that **A ⊆ B AND B ⊆ A,** then A and B are equal sets. In that case every element of A is an element of B, and every element of B is an element of A.

Example:
{7, 7, 2, 3, 1} = {1, 1, 2, 3, 7}

Difference between being an element and being a subset.
⊆ and ∈ mean completely different things.

For example {2, 5} ⊆ {5, 2, 4} but {2, 5} is not an element of {5, 2, 4}.

The only elements of {5, 2, 4} are the integers 5, 2, and 4; and {2, 5} is not the same as 5, 2, or 4.

Another example: a singleton set is not the same as its element: 4 and {4} are not the same thing. 4 ∈ {4}, but 4 ⊆ {4} is not true, and {4} ⊆ 4 is not true either.

The empty set is the set with no elements. ∅

The empty set is a subset of every set. This fact is **"vacuously" true.**
If S is a set, then ∅ ⊆ S is true. **Every element of ∅ is an element of S.**
That's a "vacuously true" statement, because the empty set has no elements.

A slightly different way of looking at this: If ∅ is **NOT** a subset of every set, then **there must be some set K such that ∅ is NOT a subset of K.** For THAT to be true, **there must be an element of ∅ that is not an element of K.** But since there are NO elements of ∅, there isn't one that's not in K.

Idea of Cartesian product, how to express ordered pairs as sets, and as "tuples"

Named after mathematician Rene Descartes, the Cartesian product of a set A and a set B is

A x B = { (a,b) | a ∈ A and b ∈ B }

(a,b) is called an ordered pair. It is not the same as the set {a,b} because (a,b) has **a first element a, and a second element b.** With sets, there's no idea of the order of the elements. For example the set {a,b} is exactly the same set as {b,a}. On the other hand, unless a=b, **(a,b) ≠ (b,a).**

When mathematicians need **to be very precise** about what they mean by an ordered pair, they can use set notation, and this definition:

(a,b) is defined as { {a}, {a,b} }

When $a \neq b$, **a is distinguished** as the 'first' element of the ordered pair **by the fact that it is in both of these sets: {a} {a,b},** and b is not in both.

There is an equality relation that has been defined that applies to ordered pairs. We say that (a,b) = (c,d) when a=c and b=d, and otherwise (a,b) ≠ (c,d)

Other problems that can be worked in class as examples:

1.2.9.a: Is $3 \in \{1, 2, 3\}$?

1.2.9.b: Is $1 \subseteq \{1\}$?

1.2.9.i: Is $\{1\} \subseteq \{1, \{2\}\}$?

Work problems with the class:

1.2.3:

a. Is $4 = \{4\}$?

b. How many elements are in the set {3,4,3,5}?

c. How many elements are in the set {1, {1}, {1, {1}}}?

1.2.5

Which of the following sets are equal

$A=\{0,1,2\}$

$B=\{x \in R \mid -1 \leq x < 3\}$

$C=\{x \in R \mid -1 < x < 3\}$

$D=\{x \in Z \mid -1 < x < 3\}$

$E=\{x \in Z^+ \mid -1 < x < 3\}$

Day Four: Wednesday, Aug 30, 2017

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* **Take Roll**

1.2.8:

A = {c, d, f, g}

B = {f, j}

C = {d, g}

Answer each question, give reasons.

a. Is $B \subseteq A$?

b. Is $C \subseteq A$?

c. Is $C \subseteq C$?

d. Is C a proper subset of A?

1.2.10:
a. Is $((-2)^2, -2^2) = (-2^2, (-2)^2)$?
b. Is $(5,-5)=(-5,5)$?
c. Is $(8-9,(-1)^{1/3}) = (-1,-1)$?
d. Is $(-2/(-4), (-2)^3) = (3/6, -8)$?

1.2.11:
A = {w, x, y, z}
B = {a,b}

Use the set-roster notation to write each of the following sets, and indicate the number of elements that are in each set:

a. A x B
b. B x A
c. A x A
d. B x B

Class work on SECTION 1.3, the LANGUAGE OF RELATIONS AND FUNCTIONS.

Topics in Section 1.3 (The Language of Relations and Functions)

* Given two sets A and B, we can **define a relation** R between some of the elements of A and some of the elements of B by specifying a subset S_R of the cartesian product A x B. It is to be understood that $(a,b) \in S_R$ means that a is related to b by the relation R (also denoted aRb).

* The ***domain*** of a relation in the cartesian product (A x B) is A.

* The **co-domain** of a relation in the cartesian product (A x B) is B.

* We make an ***arrow diagram*** of a relation contained in (A x B) by representing the elements of A as points in one region, and the elements of B by points in another region, and drawing an arrow from x to y for the pairs (x,y) that are related.

* A ***function*** F from a set A to a set B is a relation with domain A and co-domain B such that for every element $x \in A$ there is exactly one element $y \in B$ such that $(x,y) \in F$. So we see a function is a special kind of relation. Another way to say this is that, with a function relation, **every** element of the domain has to be related to something in the co-domain, and elements in the domain are **not allowed to be related to more than one** element of the co-domain. (It's like **every customer gets** 'a product' but **only one** to a customer.)

* Since each element $x \in A$ in the domain of a function F is related to exactly one element in the co-domain, we can give that element in the co-domain a name, and there will only be one possible element that the name can refer to. Often the name F(x) is used as a name for the unique element of the co-domain to which x is related.

* We can also think of a function as if it is a kind of input/output machine. The idea is that if we 'input' any $x \in A$ the function F will output $F(x) \in B$.

* In the arrow diagram of a function, every element x of the domain has exactly one arrow going from x to some element of the co-domain.

* If we graph a function from a subset of R to a subset of R, every vertical line through a point in the domain will intersect the graph at one and only one point.

Sample problems to work (pp 21-26):

1.3.1:
A={2,3,4}
B={6,8,10}
Relation R defined by (x,y) element of AxB, y/x is an integer.
a. Is 4R6?, Is 4R8?, Is (3,8) an element of R? Is (2,10) an element of R?
b. Write R as a set of ordered pairs.
c. Write the domain and the co-domain of R
d. Draw an arrow diagram for R

Day Five: Friday, Sep 01, 2017
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* **Take Roll**
* **Announcement: I assigned two more HW sets**

1.3.5:
Define a relation S from R to R as follows: For all (x,y) in RxR,
(x,y) belongs to R means x ≥ y.

a. Is (2,1) in S? Is (2,2) in S? Is 2S3? Is (-1)S(-2)?
b. Draw the graph of S in the Cartesian plane.

1.3.11: Define a relation P from R$^+$ to R as follows:

For all real x,y with x>0,
(x,y) are related if x=y^2.

Is P a function? Explain.
(If P is a function, for each x in R$^+$, there must be a unique y in R such that x=y^2)
1.3.13
Let A={-1,0,1} and B={t,u,v,w}. Define a function F:A--->B with a diagram.

-1 t

0 u

1 v

w

(diagram indicates F(-1)=F(1)=u, and F(0)=w)

a. Write the domain and codomain of F.
b. Find F(-1), F(0), and F(1).
Topics in Section 2.1 (Logical Form and Logical Equivalence)

- we start with these undefined terms: *sentence*, *true*, and *false*
- (3)(Forms of) Logical arguments
- (J1) A *statement* (aka *proposition*) is a sentence that is either true or false.
- (6) Using ~, ∨, ∧ to make compound statements ("not" "or" "and") (aka negation, disjunction, and conjunction)
- Precedence of negation over disjunction and conjunction, and co-equal precedence of disjunction and conjunction.
- (32) Expressing inequalities using "and", "or" (∧, ∨)
- (16,18) truth values of statements that are negations, disjunctions, or conjunctions
- (6) conjunction p ∧ q
- (6) disjunction p ∨ q
- Statement forms (expressions that have variables, and which become statements when the variables are replaced with statements. Example: p∧q is a statement form that becomes a statement when p is replaced by "it is hot" and q is replaced by "it is sunny")
- Evaluating compound statements in general
- Exclusive OR
- (16,18) Logical Equivalence - same truth values for all substitutions of values for variables.
- (30,32) Negations of conjunctions and disjunctions - De Morgan's Laws
- (40,41) Tautologies - always true no matter the value of variables
- (40,41) Contradictions - always false no matter the value of variables
- (48) Laws of Boolean algebra (commonly-used logical equivalences such as the commutative and associative laws, and so on)
- (48) simplifying statements using Boolean algebra

Problems to work

2.1.3. Represent the common form of each argument using letters to stand for compound sentences, and fill in the blanks so that the argument in part (b) has the same logical form as the argument in part (a).

a. This number is even or this number is odd.
This number is not even.
Therefore this number is odd.

b. ___________ or logic is confusing.
My mind is not shot.
Therefore, ______________.

J1: Which of the following is a statement?
(a) That's what I like about you.
(b) Tupelo is a kind of tree.

2.1.6,
Write the statements in symbolic form using the symbols ~, ∨, ∧ and the indicated letters to represent component statements.

s = "stocks are increasing"
i = "interest rates are steady"

a. Stocks are increasing but interest rates are steady
b. Neither are stocks increasing, nor are interest rates steady.

∀   ∈ ∉ ∃ ∄ ∍ ∧ ∨ ε ∈ ⊆ ⊂ ∪ ∩ ∞ ∅
∪ ∩ ⊆   ∅ ⊈ ≦ ≴ ≧ ≵ ≮ ≯ ∏ ∑ √ ∛ ∫ ≈ ∴ ≡

Day Six: Wednesday, Sep 06, 2017
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* **Take Roll**

2.1.16, 2.1.18,
Determine whether the statement forms are logically equivalent. In each case construct a truth table and include a sentence justifying your answer. Your sentence should show that you understand the meaning of logical equivalence.   ~, ∨, ∧

2.1.16: $p \lor (p \land q)$ versus p

How?
1. Construct truth table that has a column for truth values of $p \lor (p \land q)$ and another column for truth values of p.
2. Check each combination of truth values of p and q (and intermediate expressions) to see whether the truth value of $p \lor (p \land q)$ is always the same as the truth value of p.
   a. If in each row the truth value of $p \lor (p \land q)$ is the same as the truth value of p, then $p \lor (p \land q)$ and p are logically equivalent.
   b. If in some row the truth value of $p \lor (p \land q)$ is different from the truth value of p, then $p \lor (p \land q)$ and p are NOT logically equivalent.

2.1.18: $p \lor t$ versus t

maybe do $\sim(p \land q)$ versus $\sim p \land \sim q$,
to illustrate non-equivalence - this is an example from section 2.1.

2.1.30, 2.1.32,
Use one of DeMorgan's Laws to write a logical negation:

2.1.30: "The dollar is at an all-time high AND the stock market is at a record low."

2.1.32: $-2 < x < 7$

Day Seven: Friday, Sep 08, 2017
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* **Take Roll**

Make a truth table to prove one of DeMorgan's laws:

$\sim(p \land q) \iff (\sim p) \lor (\sim q)$
$\sim(p \lor q) \iff (\sim p) \land (\sim q)$

2.1.40, 2.1.41,
Use truth tables to figure out which are tautologies and which are contradictions:

2.1.40: (p ∧ q) ∨ (~p ∨ (p ∧ ~q))

2.1.41: (p ∧ ~q) ∧ (~p ∨ q)

2.1.48 A logical equivalence is derived below. Supply a reason for each step. (See p. 35 of Epp for the list of equivalences.)

(p ∧ ~q) ∨ (p ∧ q) ≡ p ∧ (~q ∨ q) by (a)

≡ p ∧ (q ∨ ~q) by (b)

≡ p ∧ t by (c)

≡ p by (d)

∴ (p ∧ ~q) ∨ (p ∧ q) ≡ p

Day Eight: Monday, Sep 11, 2017

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* **Take Roll**
* **Go over the last problem from Friday again, since we had to hurry through it at the end of the day.**

Topics in Section 2.2 (Conditional Statements)

* Hypotheses (antecedents), conclusions (consequents), and conditional statements
* Precedence of --> among ∨, ∧ , and ~ (--->last, ~ first)
* Truth tables for conditional statements
* Representation of if-then as OR [(p-->q) ≡ (~p ∨ q)]

MAKE SURE TO USE THE ABOVE AS THE DEFINITION OF (p-->q)

* Negation of conditional statement (~(~p ∨ q) ≡ (p ∧ ~q))
* Equivalence of a conditional statement and its contrapositive
  ((p-->q) ≡ (~q --> ~p))
  Maybe the easiest thing is to start with examples of contrapositives.
* Converse and inverse of a conditional statement [Conditional (p-->q)], [Inverse of the Conditional: (~p-->~q)], [Converse of the Conditional (q-->p)]
* Only if and the bi-conditional. In logic the phrase p only if q is logically equivalent to p --> q
* If and only if p <--> q is logically equivalent to (p-->q) ∧ (q-->p)
* Necessary and sufficient conditions
  "r is sufficient for s" means r-->s
  "r is necessary for s" means (~r --> ~s) ≡ (s-->r)
  "r is necessary and sufficient for s" means (s<-->r), which is the same meaning as s if and only if r.

Sample illustrative problems:

2.2.2 Re-write in if-then form: "I am on time for work if I catch the 8:05 bus"

Rewrite in if-then form:
J2: Bear to the right or you'll get into a collision

2.2.16 Write the two statements in symbolic form and determine whether they are logically equivalent. Include a truth table and a few words of explanation.

* **If you paid full price, you didn't buy it at Crown Books.**
* **You didn't buy it at Crown Books or you paid full price.**

2.2.19 Write the negation of "If Sue is Luiz' mother, then Ali is his cousin."

Day Nine: Wednesday, Sep 13, 2017

2.2.24 Use truth tables to establish the truth of this statement:

"A conditional statement is NOT logically equivalent to its converse."

2.2.32 Rewrite as a conjunction of two if-then statements:
"This quadratic equation has two distinct real roots if, and only if, its discriminant is greater than zero"

2.2.34 Rewrite the statement in if-then form in two ways, one of which is the contrapositive of the other.
"The Cubs will win the pennant only if they win tomorrow's game."

2.2.40 Rewrite in if-then form: "Catching the 8:05 bus is a sufficient condition for my being on time for work."

Topics in Section 2.3 (Valid and Invalid Arguments)

* An argument is a series of statements - the last statement is called the conclusion, and the others are called premises
* Valid argument - conclusion must be true if premises are true
  in other words it is impossible for the conclusion to be false when the premises are true.
* Inferred, deduced
* Testing an argument for validity with a truth table
* Critical row of a truth table
* Syllogisms(two premises + conclusion), major premise, minor premise
* Modus Ponens (p-->q; p; therefore q)
* and Modus Tollens (p-->q; ~q; therefore ~p)
* Rules of inference
* Generalization ($p \therefore p \vee q$; $q \therefore p \vee q$)
* Specialization: ($p \wedge q, \therefore p$; $p \wedge q, \therefore q$)

* Elimination (p ∨ q, ~q, ∴ p; p ∨ q, ~p, ∴ q)
* Transitivity (p-->q, q-->r, ∴ p-->r)
* Proof by division into cases (p ∨ q, p-->r, q-->r, ∴ r)
* Fallacies
* The converse error (the fallacy of affirming the consequent)
* The inverse error (the fallacy of denying the antecedent)
* Sound argument - valid and all premises true
* Unsound argument - any argument that is not sound
* Contradiction Rule (~p-->**c**, [c is a contradiction] ∴ p)

Day Ten: Friday, Sep 15, 2017

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* **Take Roll**
* **There's a new homework, and I'll assign another for today.**
*
* Sample illustrative problems:

2.3.8: Use truth tables to determine whether (p ∨ q, p-->~q, p-->r, ∴ r) is valid. Indicate which columns represent premises, and which represents the conclusion. Explain how the truth table supports your answer. Show you understand the definitions of valid/invalid.

2.3.22: Use symbols to write the logical form of the argument. Then use a truth table to test the argument for validity. Indicate which columns indicate the premises and which represents the conclusion. Include words of explanation showing you understand the meaning of validity.

If Tom is not on Team A, then Hua is on Team B.
If Hua is not on Team B, then Tom is on Team A.
∴ Tom is not on Team A, or Hua is not on Team B.

If this number is larger than 2, then its square is larger than 4.
This number is not larger than 2.
∴ The square of this number is not larger than 4.

Day Eleven: Monday, Sep 18, 2017

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* **Take Roll**

Topics in Section 3.1 (Predicates and Quantified Statements I)

* **Predicate calculus** (the symbolic analysis of predicates and quantified statements)
* **Statement calculus** (also: propositional calculus - the symbolic analysis of ordinary compound statements)
* **Predicate symbol** - obtained by removing some or all nouns from a statement (for example: "is a student at Bedford College", or "is a student at")
* **Predicate variables** - "when concrete values are substituted in place of predicate variables, a statement results." -- for example x and y are the predicate variables here: "x is a student at y".

* **Predicate** (also: propositional function or open sentence - a predicate symbol together with suitable predicate variables)
* **Domain of a predicate variable** (the set of all values that may be substituted in place of the variable)
* **Truth set of a predicate P(x)** (the set of all values x in the domain of the variable that make P(x) true - $\{x \in D \mid P(x)\}$)
* **The Universal Quantifier** ∀ (for all, for each)
* **Truth or Falsity of a Universal Statement** (true if and only if substitution of every element of the domain of the variable(s) yields a true statement)
* **The Existential Quantifier** ∃ (there exists, for some)
* **Truth or Falsity of an Existential Statement** (true if and only if the substitution of at least one value in the domain make a true statement)
* **Universal Conditional Statements** (for example: $\forall x, P(x) \rightarrow Q(x)$)
* **Equivalent Forms of Universal and Existential Statements**
* **Implicit Quantification** (for example: "if $n \in \mathbb{Z}$ then $n \in \mathbb{Q}$", or "24 can be written as the sum of two even integers") The first actually contains a universal quantification, and the second contains an existential quantification.

Sample illustrative problems:

3.1.5: Let Q(x,y) be the predicate "If $x<y$ then $x^2 < y^2$" with domain for both x and y being the set R of real numbers.

A. Explain why Q(x,y) is false if x = -2 and y=1.

B. Give values different from those in part (a) for which Q(x,y) is false.

C. Explain why Q(x,y) is true if x=3, and y=8.

D. Give values different from those in part (c) for which Q(x,y) is true.

3.1.13: Consider the following statement:

∀ basketball players x, x is tall.

Which of the following are equivalent ways of expressing this statement?

a. Every basketball player is tall.
b. Among all the basketball players, some are tall.
c. Some of all the tall people are basketball players.
d. Anyone who is tall is a basketball player.
e. All people who are basketball players are tall.
f. Anyone who is a basketball player is a tall person.

3.1.16a: Rewrite in the form "∀ ________ x, _________"
"All dinosaurs are extinct."

3.1.16c: Rewrite in the form "∀ ________ x, ________"

"No irrational numbers are integers."

3.1.16e: Rewrite in the form "∀ ________ x, ________"

"The number 2,147,581,953 is not equal to the square of any integer."

Day Twelve: Wednesday, Sep 20, 2017

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* **Take Roll**
* **Announcements:**
  - **@ We hope to pass back homework #5 and #6 by Friday, Sept 22.**
  - **@ Quiz #1 is now scheduled for Wednesday, Sept 27, on readings through section 2.3 (and through HW #6)**
  - **@ HW is assigned on section 3.1 & due next Monday, Sept 25;**

3.1.32: Let R be the domain of the predicate variable x. Which of the following are true and which are false? Give counter examples for the statements that are false.

(a) $(x>0) \Rightarrow (x>1)$

(c) $(x^2 > 4) \Rightarrow (x > 2)$

Topics in Section 3.2 (Predicates and Quantified Statements II)

* **Negating Quantified Statements:**

$(\sim(\forall\ x\in D,\ P(x)) \equiv (\exists\ x\in D,\ \sim P(x)))$

* **Relation among ∀, ∃, ∨, ∧**

When $D = \{x_1, x_2, \ldots, x_n\}$

$(\forall\ x\in D,\ P(x)) \equiv (P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n))$

and

$(\exists\ x\in D,\ P(x)) \equiv (P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n))$

* **Vacuous truth of a universal statement**

"For all balls on the table x, if x is in this (empty) bowl, x is blue"

* **Universal conditional statements**
* **Meaning of *necessary*, *sufficient*, and *only if*** in relation to quantified statements

(∀ x∈D, P(x) is sufficient for Q(x))
≡ (∀ x∈D, P(x) ---> Q(x))

(∀ x∈D, P(x) is necessary for Q(x))
≡ (∀ x∈D, Q(x) ---> P(x))

(∀ x∈D, P(x) only if Q(x))
≡ (∀ x∈D, P(x) ---> Q(x))

* **Logical equivalence of quantified statements:** identical truth values, no matter what predicates are substituted for the predicate symbols, and no matter what sets are used for the domains of the predicate values. For example both $\sim(\forall\ x\in D,\ P(x))$ and $(\exists\ x\in D,\ \sim P(x))$ have the same truth values, no matter what predicate P is, or what set D is.
* **Negation of an Existential Statement:**

~(∃ x∈D, P(x)) ≡ (∀ x∈D, ~P(x))

* **Negation of a Universal Conditional:**

~(∀ x∈D, P(x) --> Q(x))) ≡ (∃ x∈D, P(x) ∧ ~Q(x))

* **Variants of Universal Conditional Statements**

(i) Statement: (∀ x∈D, P(x) --> Q(x))
(ii) Contrapositive: (∀ x∈D, ~Q(x) --> ~P(x))
(This is logically equivalent to i)
(iii) Converse: (∀ x∈D, Q(x) --> P(x))
(This is NOT logically equivalent to i)
(iv) Inverse: (∀ x∈D, ~P(x) --> ~Q(x))
(This is logically equivalent to iii
- it is the contrapositive to iii)

Sample illustrative problems for section 3.2

3.2.3a: Write a formal negation for
(∀ fish x, x has gills.)

3.2.3c: Write a formal negation for
(∃ a movie m such that m is over 6 hours long.)

3.2.16: Write a negation for
(∀ real numbers x, if x^2 >= 1 then x >0.)

3.2.22: Write a negation for
(If the square of an integer is odd,
then the integer is odd.)

3.2.32: Write the converse, inverse, and
contrapositive of this statement:

(If the square of an integer is odd,
then the integer is odd.)

Indicate which among the four statements is true, and which is false. Give counter examples for the ones that are false.

Day Thirteen: Friday, Sep 22, 2017

Topics in Section 3.3 (Statements with Multiple Quantifiers)

* Truth of a ∀ ∃ Statement (for example, in a "Tarski World")
* Truth of a ∃ ∀ Statement (for example, in a "Tarski World")
* Key idea for the two above - imagine making "choices" in the order the quantifiers are given.
* Interpreting Multiply-Quantified Statements
Translating from Informal to Formal Language

Negations of Multiply-Quantified Statements:
Example #1
~(∀ x∈D, ∃ y∈E such that P(x,y))
≡ (∃ x∈D, ~(∃ y∈E such that P(x,y)))
≡ (∃ x∈D, ∀ y∈E, ~P(x,y))

Example #2
~(∃ x∈D, ∀ y∈E, P(x,y))
≡ (∀ x∈D, ~(∀ y∈E P(x,y)))

≡   (∀ x∈D, ∃ y∈E, ~P(x,y))

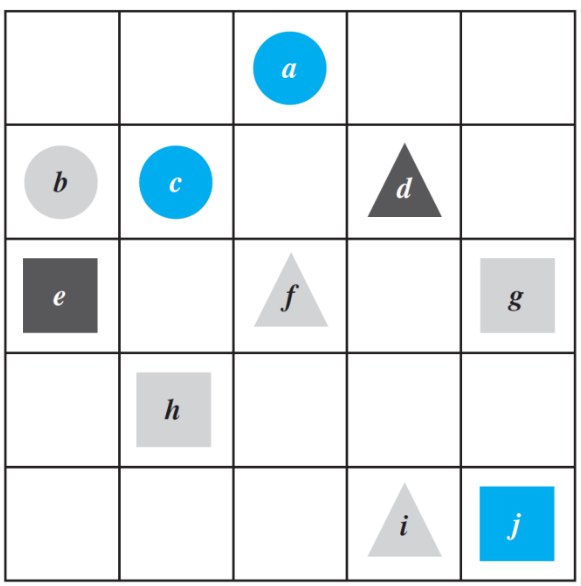
* Order of Quantifiers
    - If you interchange ∀ and ∃, usually it changes the meaning
      Example:

∀ people x, ∃ a person y such that x loves y
NOT EQUIVALENT TO
∃ people x, ∀ a person y such that x loves y

* Formal Logical Notation

Study idea: Go over the Tarski World Examples 3.3.1, 3.3.2, 3.3.9

Study idea: Do a&b of Example 3.3.10: Formalizing Statements in a Tarski WorldConsider the Tarski world shown in Figure 3.3.1.



Show that the following statement is true in this world:

For all triangles x, there is a square y such that x and y have the same color.

This one is NOT true: There is a triangle that is the same color as every square

The statement says that no matter which triangle someone gives you, you will be able to find a square of the same color. There are only three triangles, d, f, and i.

The following table shows that for each of these triangles a square of the same color can be found.

Given $x =$	**choose $y =$**	**and check that y is the same color as x.**
d	e	yes •
f or i	h or g	yes •

Sample illustrative problems for section 3.3

3.3.9a: Let D = E = {-2, -1, 0, 1, 2} Explain why the following statement is true:

∀ x∈D, ∃ y∈E such that x+y=0. (TRUE)

∃ x∈D, ∀ y∈E such that x+y=0. (NOT TRUE)

3.3.15: (a) Rewrite the statement in English, without using the symbols ∀ or ∃ or variables and expressing your answer as simply as possible, and (b) write a negation for the statement:

∀ odd integers n, ∃ an integer k such that n=2k+1.

3.3.34: (a) Rewrite the statement formally using quantifiers and variables, and (b) write a negation for the statement:

Somebody loves everybody

3.3.37: (a) Rewrite the statement formally using quantifiers and variables, and (b) write a negation for the statement:
Any even integer equals twice some integer.

Topics in Section 3.4 (Arguments with Quantified Statements)

* The rule of universal instantiation: If some property is true of everything in a set, it is true of any particular thing in the set.

* Universal Modus Ponens:
  ∀ x, If P(x) then Q(x);
  P(a) for a particular a,
  ∴ Q(a)

* Universal Modus Tollens:
  ∀ x, If P(x) then Q(x);
  ~Q(a) for a particular a,
  ∴ ~P(a)

* Validity of Arguments with Quantified Statements
* The Quantified Form of the Converse Error
  ∀ x, If P(x) then Q(x);
  Q(a) for a particular a,
  ∴ P(a) **<--- invalid conclusion**

* The Quantified Form of the Inverse Error
  ∀ x, If P(x) then Q(x);
  ~P(a) for a particular a,
  ∴ ~Q(a) **<--- invalid conclusion**

* Universal Transitivity
  ∀ x, P(x) --> Q(x)
  ∀ x, Q(x) --> R(x)
  ∴ ∀ x, P(x) --> R(x)

Sample illustrative problems for section 3.4

3.4.7: The argument may be valid by universal modus ponens or universal modus tollens. It may be invalid and exhibit the converse or inverse error. State whether valid or invalid and justify your answer:

All healthy people eat an apple a day
Keisha eats an apple a day
∴ Keisha is a healthy person

3.4.8: The argument may be valid by universal modus ponens or universal modus tollens. It may be invalid and exhibit the converse or inverse error. State whether valid or invalid and justify your answer:

All freshmen must take writing
Caroline is a freshman
∴ Caroline must take writing

Day Fourteen: Monday, Sep 25, 2017

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* **Take Roll**
* **START with a REVIEW of this on Monday, Sept 25.**

3.4.8: The argument may be valid by universal modus ponens or universal modus tollens. It may be invalid and exhibit the converse or inverse error. State whether valid or invalid and justify your answer:

All freshmen must take writing
Caroline is a freshman
∴ Caroline must take writing

*** 3.4.23: Indicate whether the argument is valid or invalid. Support your answer by drawing diagrams.**

All teachers occasionally make mistakes.
No gods ever make mistakes.
∴ No teachers are gods.

*** 3.4.24: Indicate whether the argument is valid or invalid. Support your answer by drawing diagrams.**

No vegetarians eat meat.
All vegans are vegetarian.
∴ No vegans eat meat.

Chapter Four (Elementary Number Theory and Methods of Proof)

Topics in Section 4.1 (Proof and Counterexample I: Introduction)

* Definitions of even and odd integers (2k or 2k+1)
* Definition of a prime number: (p>1 in Z s.t. if n,m in Z^+ and nm=p, then n=p or m=p)
* Definition of a composite number: (c>1 in Z, and c=nm for integers c>n>1 and c>m>1)
* Constructive Proofs of Existential Statements (to show ∃ x∈D s.t. Q(x), one can either find an x

that makes Q(x) true, or give a set of directions for finding an x that makes Q(x) true. Either way, that's a constructive proof.)

* Non-Constructive Proof of Existence: (a) show existence is guaranteed by some theorem, or (b) show that the assumption that there is no x that makes Q(x) true leads to a contradiction.
* Disproving Universal Statements by Counterexample (Show P(x) ---> Q(x) is false by providing a counter example c such that P(c) && ~Q(c))
* The Method of Exhaustion (show P(x) ---> Q(x) by individually checking each x such that P(x) is true.)
* Generalizing from the Generic Particular (e.g. "Direct Proof": show P(x) ---> Q(x) by assuming x is some generic element of the domain, and basing the demonstration only on that)
* Existential Instantiation - if you have established that something exists, you can give it a name in your logical arguments, so long as you don't give it a name that is already being used for something else. (Example: if we know that m is an even number, then we know it is twice some integer, so we can give that integer the name k, and write m = 2k.)

Sample illustrative problems for section 4.1

4.1.8: Prove there is a real number $x>1$ such that $2^x > x^{10}$.

The Homework problem (#4.1.10) is to prove that there is an integer n such that

$2n^2 - 5n + 2$

is prime. Hint:

$2n^2 - 5n + 2$ can be factored as

$2n^2 - 5n + 2 = (n-2)*(2n-1)$

The easiest thing may be to just start out trying a few values of n in the formula, starting at 0. (If it interests you, after you find one solution, look for one more.)

Day Fifteen: Wednesday, Sep 27, 2017

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* **Quiz #1**

Day Sixteen: Friday, Sep 29, 2017

4.1.30: Prove the statement. Use only the definitions of the terms and the "Assumptions" listed on page 146, not any previously established properties of odd and even integers. Follow the directions in this section for writing proofs of universal statements.

For all integers m, if m is even then 3m+5 is odd.

4.1.35: Prove the statement is FALSE:

There exists an integer m >= 3 such that (m^2-1) is prime.

4.1.39: Find the mistake in the "proof":

Theorem: The difference between any odd integer and any even integer is odd.

"Proof: Suppose n is any odd integer, and m is any even integer. By definition of odd, n=2k+1, where k is an integer, and by definition of even, m=2k, where k is an integer. Then n-m = (2k+1)-2k = 1. But 1 is odd. Therefore, the difference between any odd integer and any even integer is odd."

4.1.50: Determine whether the statement is true or false. Justify your answer with a proof or counter-example, as appropriate. Use only the definitions of the terms and "Assumptions" listed on page 146, not any previously established properties.

For all integers n and m, if n-m is even then n^3-m^3 is even.

4.1.55: Determine whether the statement is true or false. Justify your answer with a proof or counter-example, as appropriate. Use only the definitions of the terms and "Assumptions" listed on page 146, not any previously established properties.

Every positive integer can be expressed as a sum of three or fewer perfect squares.

Day Seventeen: Monday, Oct 2, 2017

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* **Go over answers to Quiz #1**

Start with a review of this on Monday, Oct 2

4.1.55: Determine whether the statement is true or false. Justify your answer with a proof or counter-example, as appropriate. Use only the definitions of the terms and "Assumptions" listed on page 146, not any previously established properties.

Every positive integer can be expressed as a sum of three or fewer perfect squares.

Day Eighteen: Wednesday, Oct 4, 2017

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* **Take Roll**
* **Handback tests and HW**
* **There are more HW assignments now. Please have a look**

Topics in Section 4.2 (Proof and Counterexample II: Rational Numbers)

* **Rational Numbers (Q)**

$Q = \{ r \in R \mid r = a/b, \text{ for integers } a,b, \text{ with } b \neq 0\}$

The elements of Q are called rational because they are ratios of integers

* More on Generalizing from the Generic Particular - seen as being prepared to meet the challenge of showing P(x) is true for any particular value of x that is offered by an 'adversary'.

* Deriving New Mathematics from Old - Once you have proved something, you can use it in proofs of new things.

* A Corollary is a statement whose truth can be immediately deduced from a theorem that has already been proved.

Sample illustrative problems for section 4.2

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Directions: Is the statement true or false? If true prove from definitions. If false, try to find a minor change that makes it true, and then prove the new statement.

*** 4.2.H16: The quotient of any two rational numbers is rational.**

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*** 4.2.39: Find the flaw in this attempted proof that the sum of two rational numbers is rational:**

"PROOF: Suppose r and s are rational numbers. If r+s is rational then by definition of rational r+s = a/b where a,b are integers and b≠0. Also, since r and s are rational, r=i/j, s=m/n for integers i,j,m,n with j≠0 and n≠0. It follows that r+s = (i/j) + (m/n) = (a/b), which is a quotient of two integers, with a non-zero denominator. Hence it is a rational number. This was what was to be shown."

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Topics in Section 4.3 (Direct Proof and Counterexample III: Divisibility)

* When n∈Z, and d∈Z with d≠0, d|n means "d divides n," which means that n=dm, where m∈Z. For example 2|10 because 10=2*5. We also express this idea by saying "n is a multiple of d," "d is a factor of n," and "d is a divisor of n."

* Divisors of zero: If d∈Z with d≠0, we say d is a "divisor of 0" because it is true that 0=d*0. (Every non-zero integer is divisor of 0.) For example 0=42*0, and 42≠0, so 42 is a divisor of 0.

* However 0 is not a divisor of anything. (Part of the definition of a divisor d is that d≠0.

* Theorem 4.3.1 - A Positive Divisor of a Positive Integer: For all integers m, n, if m and n are positive and m|n, then m≤n.

(Idea of the proof: n=mk, k∊Z. k must be ≥0. ∴ k≥1, so mk≥m, i.e. n≥m, qed)

* Theorem 4.3.2 - Divisors of 1: The only divisors of 1 are 1 and -1.

(Idea of the proof: 1*1=1 and (-1)*(-1)=1 shows that 1 and -1 are divisors of 1. If m is any divisor of 1, then 1=mk, m∊Z and k∊Z. If m and k are positive then by Theorem 4.3.1, 1≥m. The only positive integer ≤1 is 1, so that means m is 1. If m and k are not both positive, they must both be negative. In that case, 1=(-m)(-k), where both -m and -k are positive. Again by Theorem 4.3.1, we can conclude that -m is 1, in other words m=-1.

* Theorem 4.3.3 - Transitivity of Divisibility: ∀ k∊Z, m∊Z, n∊Z, if k|m and m|n, then k|n

(Idea of proof: m=kd, n=mh, where d∊Z and h∊Z. ∴ n=mh=(kd)h=k(dh). dh∊Z, so we have shown that k|n. qed)

* Theorem 4.3.4 - Divisibility by a Prime: Any integer n>1 is divisible by a prime number.

(Idea of proof: If n is prime, then it is divisible by a prime number - itself. If n is not prime then n=r_0s_0 where r_0 and s_0 are both integers "properly" between 1 and n. r_0 divides n. If r_0 is prime, then n is divisible by a prime. If r_0 is not prime then $r_0=r_1s_1$, where r_1 and s_1 are both integers "properly" between 1 and r_0. By the transitivity of divisibility, r_1|n. Also $1<r_1<r_0<n$. If r_1 is prime, then n is divisible by a prime. If not, r_1 can be factored as $r_1=r_2s_2$, where r_2 and s_2 are both integers "properly" between 1 and r_1, r_2|n (by transitivity of divisibility), and $1<r_2<r_1<r_0<n$. If r_2 is prime, then n is divisible by a prime. This process of finding integers r_i has to stop with finding an r_i that is prime eventually, because if it did not, there would be an infinite <u>descending</u> sequence of integers greater than 1: $n>r_0>r_1>r_2>r_3>$ ... > 1. This is a contradiction because, whatever positive integer n is, there are only finitely many integers between n and 1. Since the process stops with an r_i that is prime, and since that r_i divides n (by transitivity of divisibility), we have proved that n is divisible by a prime. qed)

* Theorem 4.3.5 - Unique Factorization of Integers Theorem (The Fundamental Theorem of Arithmetic): Given any integer n>1, n can be expressed as the product of a list of prime numbers. (In this kind of list, the same prime is allowed to appear multiple times.) Except for different possible orderings, there is only one such list for each n.

  Example: 24 = 2*2*2*3, and there is no other way to write 24 as a product of primes, except for shuffling the factors around like this: 24=2*2*3*2.

Sample illustrative problems for section 4.3

* **4.3.15: Prove directly from the definition of divisibility:**
  For all integers a, b, and c, if $a|b$ and $a|c$, then $a|(b+c)$

Day Nineteen: Friday, Oct 6, 2017

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* **Take Roll**

* **4.3.H19: Determine whether the statement is true or false. If true, prove directly from the definitions. If false, give a counter example.**

  For all integers a, b, and c, if a divides b then a divides bc.

* **4.3.20: Determine whether the statement is true or false,. If true, prove directly from the definitions. If false, give a counter example.**

  The sum of any three consecutive integers is divisible by 3. (Integers $m<n$ are consecutive if and only if $n=m+1$.)

* **4.3.31: Determine whether the statement is true or false. If true, prove directly from the definitions. If false, give a counter example.**

  For all integers a and b, if $a|10b$, then $a|10$ or $a|b$.

Topics in Section 4.4 (Proof and Counterexample IV: Division into Cases and the Quotient-Remainder Theorem)

* The basic **idea of dividing** an integer j by another integer k is to 'represent j as some groups of size k'.

* For example, the idea of dividing 13 by 5 is to express 13 as two groups of five, with three left over.

  11111   111   = 13 = 2*5+3

* Theorem 4.41 - The **Quotient Remainder Theorem:** Given any integer n and positive integer d, there exist unique integers q and r such that

  $$n = (d * q) + r \quad \text{and } 0 <= r < d$$

Example:
53 = 3 * 17 + 2;
So here q=17 and r=2 $(0 <= r < d)$

-53 = 3 * (-17) - 2 = 3 * (-18) + 1;
So here q=(-18) and r=1 $(0 <= r < d)$

When d is a factor of n, n=d*k for some integer k, and the remainder r=0.

In this case, -n=d*(-k) and the remainder is also zero.

When n=d*k+r, with 0 < r < d (in other words r≠0)

then the quotient remainder numbers for -n are:

-n = d*(-k-1) + (d-r)

The example above of 53=3*17+2 and -53 =3*(-18)+1 illustrate the idea.

When n>0 and d>0, and n=dk+r (0<=r<d), there are programming language expressions for the quotient and remainder that use the C++ (or Java) / and % operators.

k=(n/d)=(n div d) and r=(n%d)=(n mod d)

*** We can use Theorem 4.4.1 (The Quotient-Remainder Theorem) to prove that every integer is either even or odd.**

(Idea of Proof: n=2q+r, where 0<=r<2. r must be 0 or 1. Therefore n is either even or odd. The uniqueness of q and r implies that n cannot be both even and odd.)

* Method of **Proof by Division into Cases**
To prove a statement of the form

"If A_1, or A_2, or ... or A_n, then C,"

prove all of

"If A_1 then C, if A_2 then C, ... and if A_n then C."

The process shows that C is true regardless of which of A_1, A_2, ..., A_n happens to be the case.

An example of this is a proof that two consecutive integers have opposite parity. It's convenient to divide the proof into the two cases where the first integer is even, and the case where the first integer is odd.

* **Theorem 4.4.6 - The Triangle Inequality:** Let x, y be any two real numbers. |x+y|<=|x|+|y|.

* **Lemma 4.4.4:** For any real number r, -|r|<=r<=|r|.

Proof: if r>=0 then the inequalities just say that -r<=r<r, which is obviously true. On the other hand if r<0, then the inequalities just say that r<=r<=-r, which is also obviously true.

[
Idea of the proof of the triangle inequality: From the lemma, we know that x<=|x| and y<=|y| are true. Therefore x+y <= |x|+|y|.

Case 1: x+y >= 0. In this case |x+y| = x+y. Since x+y <= |x|+|y|, we can conclude |x+y| <= |x|+|y| in case 1.

Case 2: $x+y < 0$. In this case
$|x+y| = -(x+y) = (-x) + (-y)$.
We know from the lemma that $(-x) <= |-x| = |x|$, and
$(-y) <= |-y| = |y|$.

Therefore $(-x) + (-y) <= |x|+|y|$.

Since, in this case $|x+y| = (-x) + (-y)$, we can
also conclude that $|x+y| <= |x|+|y|$ in case 2.
]

Day Twenty: Monday, Oct 9, 2017

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* **Take Roll**

Sample illustrative problems for section 4.4

* **Problem 4.4.23: Prove that for all integers n, if n mod 5 = 3, then n^2 mod 5 = 4.**

(In other words if the remainder upon division of n by 5 is 3, then the remainder upon division of n^2 by 5 is 4.)

* **Problem 4.4.51: If m, n, a, b, and d are integers, d>0 and m mod d = a and n mod d = b, is (m+n) mod d = (a+b)? Is (m+n) mod d = (a+b) mod d? Prove your answers.**

A proof of the UNIQUENESS of the quotient and the remainder in the Quotient-Remainder Theorem.

Suppose $n = dh + r = dk + s$, where n,d,h,k,r,s are integers, $d>0$, $0 \le r<d$, and $0 \le s<d$.

One may prove the uniqueness constructively like this:

Suppose that $n = dh+r = dk+s$, where $0 \le r<d$ and $0 \le s<d$.

Without loss of generality, we may assume that the names h and k are chosen so that, $h \ge k$.

Solving $dh+r = dk+s$ for s, we get $r+d(h-k)=s$
Since $0 \le r$, we can write

$d(h-k) = 0 + d(h-k) \le r + d(h-k) = s < d$

and therefore, by transitivity, we can conclude that

$d(h-k) < d$

We can cancel $d>0$ from the expression above, which gives us

$(h-k) < 1$.

Since $h \ge k$, $(h-k) \ge 0$ is a non-negative integer. Since we have just shown that $(h-k)<1$, we conclude that the only value it can have is zero. Therefore $(h-k)=0$ and $h=k$.

If we now go back and look at this relation

dh+r = dk+s,

and substitute h for k, we get

dk+r = dk+s

If we subtract dk from both sides we see that r=s.

This completes the proof of the uniqueness of the quotient and the remainder in the Quotient-Remainder Theorem.

We do not cover Section 4.5 in this course.

Topics in Section 4.6 (Indirect Argument: Contradiction and Contraposition)

* The Method of 'Proof By Contradiction' (this method uses the 'rule of the excluded middle' - the idea that if a statement is NOT FALSE, then it must be TRUE, because there's presumably nothing 'in between' TRUE and FALSE.)
  1. If you want to prove S by contradiction, begin by making ~S, the logical negation of S, a PREMISE.
  2. Make logical deductions that lead to a contradiction - something that is known to be FALSE.
  3. Observe that ~S must be FALSE, because to assume it is true leads to a FALSE conclusion. Observe that S must therefore be TRUE. (~S is FALSE means "S is TRUE")

Example: Let $\{p_1, p_2, ..., p_n\}$ be a finite list of $n>=1$ prime numbers, where n is an integer.
Let $q = p_1*p_2*...*p_n + 1$,

in other words q is one more than the product f the list of primes.

Prove this statement S:
"None of the primes in the list $\{p_1, p_2, ..., p_n\}$ is a divisor of q."

Proof: Assume ~S is true. In other words assume that there is a prime p in the list such that $p|q$, i.e. q=pk, where k is an integer.

Note also that p divides $p_1*p_2*...*p_n$ - in other words $p_1*p_2*...*p_n = ph$ where h is the product of all the primes in the list that are not equal to p. (If p is the only prime in the list, then h=1.) Since

$q = p_1*p_2*...*p_n + 1$,

$1 = q-p_1*p_2*...*p_n = pk-ph = p(k-h)$.

This shows that 1 is a multiple of p.

But Theorem 4.3.2, which was proved in section 4.3, says "The only divisors of 1 are 1 and -1."

Since p is a prime and $p\neq 1$, we have reached a <u>contradiction</u> by making the assumption that one of the primes p in the list is a divisor of q. Therefore the assumption must be <u>false</u>.

Therefore q is not divisible by any of the primes in the list. This completes the proof.

Sample illustrative problems for section 4.6

*** Carefully formulate the negations of the statement. Then prove the statement by contradiction.**

4.6.11: S = "The product of any nonzero rational number and any irrational number is irrational."

Day Twenty-One: Friday, Oct 13, 2017

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* Take Roll
* (More homework assignments are now in place - for the next two weeks.)
* The next quiz is tentatively scheduled for November 6.

*** Prove by contraposition:**
[The idea is to prove a statement of the form P(x) ==> Q(x) by proving the logically equivalent statement ~Q(x) ==> ~P(x)]

4.6.19: S = "If a product of two positive real numbers is greater than 100, then at least one of the numbers is greater than 10."

Topics in Section 5.1 (Sequences)

* Definition of a sequence

A function F:D-->S from a set D to a set S, where D is either an 'integer interval' or an 'ascending integer ray'. In other words, the set D is either of the form $D = \{i \in Z | i \geq n\}$, where n is some integer, or of the form $D = \{i \in Z | m \geq i \geq n\}$, where m and n are integers, with $m \geq n$.

Example:
$a_i = (-1)^i/(i!)$, $i \geq 1$
$= -1, 1/2, -1/6, 1/24, -1/120, \ldots$

* Definition of term, subscript, index, initial term, final term, subscript notation, definition of an ellipsis [...], infinite sequence, explicit formula, general formula (formulae for a general term of the sequence, expressed as a function of the subscript)

* An alternating sequence ... See the previous example.

* Finding an explicit formula to fit given initial terms

* Summation Notation (See notation in the inset for Theorem 5.1.1)

* Expanded form, index of a summation, lower limit, upper limit
* Computing summations
* Summation terms given by a formula
* Changing between summation form and expanded form

* Separating off terms
* Telescoping sums
* Product notation (example below)

* Theorem 5.1.1: Properties of Summations and Products

Theorem 5.1.1

If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and c is any real number, then the following equations hold for any integer $n \geq m$:

1. $\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$

2. $c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$ generalized distributive law

3. $\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \cdot b_k)$.

* Change of variable -

(**Example** - the sum of k^3 from k=1 to k=3 is the same as the sum of j^3 from j=1 to j=3). [sums of consecutive cubes: 1, 9, 36, 100, ...]

(**Another example** of change of variable: the sum of 1/(k+1) from k=1 to k=n is the same as the sum of 1/h from h=2 to h=n+1.)

* Factorial and binomial coefficient ("n choose r") notation
* Sequences in Computer Programming
* Dummy variable in a loop
* Applications

Day Twenty-Two: Monday, Oct 16, 2017

*** Take Roll**

Sample illustrative problems for section 5.1

*** Write the first four terms of the sequences defined by the formula**

5.1.3: $c_i = [(-1)^i]/3^i$ for all integers i≥0

*** Write the first four terms of the sequences defined by the formula**

5.1.5: $e_n = \lfloor n/2 \rfloor \cdot 2$ (2*floor(n/2) ∀ integers n≥0

*** Find an explicit formula for the sequence**

5.1.12: 1/4, 2/9, 3/16, 4/25, 5/36, 6/49

*** Write using summation or product notation**

5.1.46:

(2/(3*4)) - (3/(4*5)) + (4/(5*6)) -(5/(6*7)) + (6/(7*8))

*** Write using summation or product notation**

5.1.50:
1/2! + 2/3! + 3/4! + ... + n/(n+1)!

Topics in Section 5.2 (Mathematical Induction I)

* **The Principle of Mathematical Induction**

Let c be a fixed integer. Let P(n) be a statement that is defined for all integers $n \geq c$. Suppose the following two statements are true:

1. P(c) is true.
2. For all integers $k \geq c$,
   if P(k) is true, then P(k+1) is true.

If so, then P(n) is true for all integers $n \geq c$

* **A principle logically equivalent to the principle of mathematical induction:**

Let c be a fixed integer. Suppose S is any set of integers satisfying

1. $c \in S$
2. for all $k \geq c$ if $k \in S$ then $k+1 \in S$

Then S must contain every integer greater than or equal to c.

* (S is the set of all integers for which the theorem is true, according to the principle of mathematical induction)

* **The basis step** (aka base case) - proving P(c)

* **The inductive hypothesis** - the assumption that P(k) is true, $\exists\, k \in Z$, $k \geq c$

* **The inductive step** - proving P(k) --> P(k+1)

* **A *closed form*** is a formula for the value of a sum that does not contain an ellipsis or a summation sign.

  Example: n(n+1)/2 is a closed form for 1 + 2 + ... + n.

Day Twenty-Three: Wednesday, Oct 18, 2017

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* **Take Roll**
* **Restart the problem below on Wednesday**

Sample illustrative problems for section 5.2

* **5.2.10 Prove the statement by mathematical induction:**

$1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

for all integers $n \geq 1$

* **5.2.13 Prove the statement by mathematical induction:**

The sum from i=1 to n-1 of i(i+1) equals n(n-1)(n+1)/3 for integers $n \geq 2$

* **5.2.20 Use the formula for the sum of the first n integers and/or the formula for the sum of a**

geometric sequence to evaluate the sum, or to write it in closed form:

$4 + 8 + 12 + 16 + ... + 200$

* 5.2.22 Use the formula for the sum of the first n integers and/or the formula for the sum of a geometric sequence to evaluate the sum, or to write it in closed form:

$3 + 4 + 5 + 6 + ... + 1000$

Day Twenty-Four: Friday, Oct 20, 2017
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* **Take Roll**

* 5.2.27 Use the formula for the sum of the first n integers and/or the formula for the sum of a geometric sequence to evaluate the sum, or to write it in closed form:

$5^3 + 5^4 + 5^5 + ... + 5^k$,
where k is any integer with $k \geq 3$.

Topics in Section 5.3 (Mathematical Induction II)

* Proving divisibility of $2^{2n} -1$ by 3

* Proving the inequality $2n+1 < 2^n$, for $n \geq 3$

* Proving a property of a sequence, for example, when $a_1 = 2$, and $a_k = 5a_{k-1}$
  The problem is to show that $a_n = 2(5^{n-1})$ for $n \geq 1$.

* "A Problem with Trominoes"

Sample illustrative problems for section 5.3

* 5.3.6: For each positive integer n, let P(n) be the property (statement)

5^n-1 is divisible by 4.

a. Write P(0). Is P(0) true?
b. Write P(k).
c. Write P(k+1)
d. In a proof by mathematical induction that this divisibility property holds for all integers $n \geq 0$, what must be shown in the inductive step?

* 5.3.10: Prove this statement by mathematical induction:

n^3-7n+3 is divisible by 3 for each integer $n \geq 0$.

* 5.3.21: Prove this statement by mathematical induction:

$\text{sqrt}(n) < \text{sqrt}^{-1}(1) + \text{sqrt}^{-1}(2) + ... + \text{sqrt}^{-1}(n)$;
for all integers $n \geq 2$

Day Twenty-Five: Monday, Oct 23, 2017

Topics in Section 5.4 (Strong Mathematical Induction and the Well-Ordering Principle for the Integers)

* **Principle of Strong Mathematical Induction**

Let P(n) be a property that is defined for integers n, and let **a** and **b** be fixed integers with **a≤b.** Suppose the following two statements are true:

1) P(a), P(a+1), ... and P(b) are all true (basis step)
2) For any integer k≥b, if P(i) is true for all integers i from **a** through **k,** then P(k+1) is true. (inductive step)

Then the statement

for all integers n≥a, P(n)

is true. (The supposition that P(i) is true for all integers i from a through k is called the inductive hypothesis. Another way to state the inductive hypothesis is to say that P(a), P(a+1), ... P(k) are all true.)

* Actually the Principle of Strong Mathematical Induction is not really any 'stronger' than the Principle of Mathematical Induction - anything that can be proved with one can also be proved with the other. However sometimes one or the other is more convenient for someone constructing a proof.

* **The Principle of Strong Mathematical Induction** is also known as **the second principle of induction,** the second principle of finite induction, and the principle of complete induction.

* Using the Principle of Strong Mathematical Induction, we can conveniently prove that every integer greater than 1 is divisible by a prime.

* Another example is a proof of a property of a sequence

* Another example is proof that a multiplication of k factors always requires k-1 multiplications, regardless of how the factors are associated.

* Another example: Existence and uniqueness of binary integer representations

* The Well Ordering Principle is equivalent to both the ordinary and the strong principles of mathematical induction.

The Well Ordering Principle:

Every non-empty set of positive integers contains a least element.

* Equivalently, *every non-empty set of integers that is bounded below contains a least element.*

* Application: The proof-of-existence part of the Quotient-Remainder Theorem.

(S = set of all non-negative integers of the form n-dk, where k is an integer)

Here is a proof of the existence part of the Quotient-Remainder Theorem that uses only ordinary mathematical induction.

Theorem: ∀ n ∈ Z, ∀ d, 1≤d∈Z, ∃ q∈Z, ∃ r∈Z, such that P(n) is true, where P(n) is this statement: n=dq+r, and 0≤r<d

Proof: Case #1, assume n≥0. We prove case #1 by induction

(1) 0 = d*0+0 establishes that P(0) is true with q=0, and r=0

(2) Suppose that P(k) is true, ∃ 0≤k∈Z.

Then k = dh+s, where h∈Z, s∈Z, and 0≤s<d.
It follows that k+1=dh+(s+1).

We know that 0<s+1≤d. If s+1<d, then k+1=dh+(s+1) establishes that P(k+1) is true. If it is not true that s+1<d, then s+1=d. In that case, this is true: k+1=dh+d=d(h+1), which shows that P(k+1) is true with q=h+1, and r=0.
We have demonstrated that P(k) --> P(k+1).
It follows from the principle of mathematical induction that P(n) is true for all integers n≥0.

Case #2: Assume n<0. By the proof above, P(-n) is true. Therefore

-n = dp+t, where p∈Z, t∈Z, and 0≤t<d

Therefore n = d(-p)-t, and (-d)<(-t)≤0

If t=0, then this shows that P(n) is true with q=-p, and r=0.

If t≠0, then (-d)<(-t)<0, and writing n this way

n = d(-p-1)+(d-t),

we observe that (-d)<(-t)<0 implies 0<(d-t)<d, which shows P(n) to be true with q=(-p-1) and r=(d-t).

This establishes that P(n) is true for integers n<0. So cases #1 and #2 combined prove that P(n) is true for all integers n, qed.

* Application: Proof that a strictly decreasing sequence of non-negative integers is finite.

Sample illustrative problems for section 5.4

*** 5.4.7: Suppose that a sequence is defined as follows.**
g(1)=3, g(2)=5,
g(k)=3g(k-1)-2g(k-2), ∀k, 3≤k∈Z

Prove: $g(n)=2^n+1$, ∀n, 1≤n∈Z

*** 5.4.8a: $h(0)=1$, $h(1)=2$, $h(2)=3$,
$h(k)=h(k-1)+h(k-2)+h(k-3)$,
$\forall k,\ 3\le k\in Z$**

Prove: $h(n)\le 3^n$, $\forall n,\ 0\le n\in Z$

Day Twenty-Six: Wednesday, Oct 25, 2017

Topics in Section 5.6 (Defining Sequences Recursively)

* A recurrence relation for a sequence

$a_0, a_1, a_2, \ldots$

is a formula that relates each term

a_k

to certain of its predecessors

$a_{k-1}, a_{k-2}, \ldots, a_{k-i}$,

where i is an integer with $k-i\ge 0$.

The initial conditions for such a recurrence relation specify the values of

$a_0, a_1, a_2, \ldots, a_{i-1}$,

if i is a fixed integer, or

$a_0, a_1, \ldots, a_m$,

where m is an integer with $m\ge 0$, if i depends on k.

* Computing Terms of a recursively defined sequence

For example:
$f(1)=1$, $f(2)=1$, and $f(n)=f(n-1)+f(n-2)\ldots$ for $n\ge 2$.

Calculate $f(3)$, $f(4)$ and $f(5)$.

* Writing a recurrence relation in more than one way

For Example:
Description #1: $f(1)=1$, $f(2)=1$, $f(n)=f(n-1)+f(n-2)$
Description #2: $h(0)=1$, $h(1)=1$, $h(k+1)=h(k)+h(k-1)$

* (Different) Sequences that satisfy the same recurrence relation

For example, the Fibonacci sequence satisfies
$f(n) = f(n-1)+f(n-2)$ for $n\ge 2$

The Lucas sequence satisfies the same relation.
However the initial conditions are different.
Fibonacci: $f(0)=1$, $f(1)=1$
Lucas: $f(0)=1$, $f(1)=3$

* Showing that a sequence given by an Explicit Formula Satisfies a Certain Recurrence Relation

For example, the proof of the recursion relation
C(k) = C(k-1)(4k-2)/(k+1) for the Catalan
numbers, which have this closed form:

[C(n) = (2n choose n) * (1/(n+1))

* Examples of Recursively Defined Sequences
* The *recursive paradigm* (aka the *recursive leap of faith - "You suppose you know the solutions to smaller subproblems and ask yourself how would you make best use of that knowledge to solve the larger problem."*)
* The Tower of Hanoi
* The Fibonacci Numbers
* Compound Interest
* Recursive Definitions of Sum and Product

Sample illustrative problems for section 5.6

*** 5.6.3: Find the first four terms**
$c_k = k(c_{k-1})^2$, $k\geq 1$; $c_0=1$

*** 5.6.8: Find the first four terms**
$v_k = v_{k-1} + v_{k-2} + 1$, $k\geq 3$;
$v_1 = 1$; $v_2 = 3$

*** 5.6.11: $c_n = 2^n - 1$, $0\leq n$, n in Z.**
Show $c_k = 2c_{(k-1)} + 1$ for $1\leq k$, k in Z

*** 5.6.13:**
(A) $t_n = 2+n$, $n\geq 0$;
Show that $t_k = 2t_{(k-1)} - t_{(k-2)}$, $k\geq 3$

Day Twenty-Seven: Friday, Oct 27, 2017

Topics in Section 5.7
(Solving Recurrence Relations by Iteration)

* Given a sequence that satisfies a ***recurrence relation***, a ***solution*** to the sequence is an ***explicit formula*** for each member of the sequence.

  Example: The Fibonacci sequence satisfies this recurrence relation:

$F_1=1$, $F_2=1$, $F_n = F_{n-1} + F_{n-2}$, $\forall n$, $2\leq n\in Z$

This is a solution (explicit formula):

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$$

where:

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61803\,39887\cdots$$

* The Iteration Method of finding a solution:
  (1) Generates successive terms of the sequence until you see a pattern;
  (2) Guess an explicit formula;
  (3) Verify the formula

Example: $a_0=1$, $a_n = a_{n-1} + 2n + 1$, $\forall n$, $1 \le n \in Z$
Solution by iteration:
$a_1 = a_0 + 2(1) + 1 = 1 + 3 = 4 = 2^2$,
$a_2 = a_1 + 2(2) + 1 = 4 + 5 = 9 = 3^2$,
$a_3 = a_2 + 2(3) + 1 = 9 + 7 = 16 = 4^2$,
$a_4 = a_3 + 2(4) + 1 = 16 + 9 = 25 = 5^2$

So it looks like $a_n=(n+1)^2$ is an explicit formula for a_n.

The next step is usually to attempt to prove the guessed formula is correct, typically with proof by induction.

* Arithmetic sequence: $\{a_0, a_1, a_2, \ldots\}$, $\exists d \in R$, $a_n=a_{n-1}+d$, $\forall n$, $1 \le n \in Z$

(explicit formula solution: $a_n=a_0+dn$, $\forall n$, $0 \le n \in Z$)

* Geometric sequence: $\{a_0, a_1, a_2, \ldots\}$, $\exists d \in R$, $a_n=da_{n-1}$, $\forall n$, $1 \le n \in Z$

(explicit formula solution: $a_n=a_0 d^n$, $\forall n$, $0 \le n \in Z$)

* other formulas:

Arithmetic Series: $1+2+3+\ldots+n = n(n+1)/2$

Geometric Series: $1+x+x^2+\ldots+x^n = (x^{n+1}-1)/(x-1)$; $1 \ne x \in R$

Sample illustrative problems for section 5.7

*** 5.7.1b: For integers $n \ge 1$,**

(A) $1+2+3+\ldots+n = n(n+1)/2$

Use that fact to find a formula for the expression

$3+2+4+6+8+\ldots+2n$; $\forall n$, $1 \le n \in Z$

*** 5.7.6: Use iteration to guess an explicit formula:**

$d_1=2$; $d_k=2d_{k-1}+3$, $\forall k$, $2 \le k \in Z$

*** 5.7.10: Use iteration to guess an explicit formula:**

$h_0=1$; $h_k=2^k - h_{k-1}$, $\forall k$, $1 \le k \in Z$

Day Twenty-Eight: Monday, Oct 30, 2017
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* **Do Roll**
* **Do some review of the previous problem**

*** 5.7.35: Use mathematical induction to verify the correctness of the formula derived in exercise 5.7.10.**

Topics in Section 5.8 (Second-Order Linear Homogeneous Recurrence Relations with Constant Coefficients)

* A second-order linear homogeneous recurrence relation with constant coefficients has the form

$$x_n = Ax_{n-1} + Bx_{n-2},$$

for all n greater than some fixed integer,

where A and $B\neq 0$ are fixed constant real numbers.

It's "second-order" because the value farthest back in the sequence used to define x_n is x_{n-2}.

It's "linear" because each term contains no more than one of x_n, x_{n-1}, or x_{n-2}: there are no second or higher powers of x_n, x_{n-1}, or x_{n-2}, and no products of two or more of x_n, x_{n-1}, or x_{n-2}.

It's "homogeneous" because every term has the same "degree" - the same total number of powers of any of x_n, x_{n-1}, or x_{n-2}. (When a recurrence relation is linear, the only way it can fail to be homogeneous is if it has a constant term.)

(Incidentally, people commonly make the error of using the word "homogenous" when they mean "homogeneous." We will stick with the more proper term: "homogeneous.")

It has "constant coefficients" because A and B are fixed constants that do not depend on n.

* If t is some number such that

$\mathbf{t^2=At+B}$,

then

$t^k=At^{k-1}+Bt^{k-2}$

for all integers $k\geq 2$, which can be proved just by multiplying through the first equation by t multiple times.

* The equation $t^2=At+B$, and equivalently $\mathbf{t^2-At+B=0}$, is called the characteristic equation (also auxiliary equation) of the relation. Lemma 5.8.1 says basically that the sequence

$\{x_0=1,\ x_1=s,\ x_2=s^2,\ x_3=s^3,\ x_4=s^4\ ,\ ...\ \}$

satisfies the relation

$x_n = Ax_{n-1} + Bx_{n-2}$

if and only if s is a root of the characteristic equation.

Example: Consider the recurrence relation
$x_n = x_{n-1} + 2x_{n-2}$

The characteristic equation is $T^2-T-2=0$, which factors as $(T-2)(T+1)=0$. So the solutions are $T=2$ and $T=-1$. By lemma 5.8.1, the sequences

$\{1,2,2^2,2^3,...\}$ and $\{1,-1,(-1)^2,(-1)^3,...\}$ satisfy the recurrence, and are the ONLY sequences of powers that satisfy the recurrence.

* Any 'linear combination' of solutions to the recurrence $x_n = Ax_{n-1} + Bx_{n-2}$ is also a solution.

Example: We showed that these sequences $\{1,2,2^2,2^3,...\}$ and $\{1,-1,(-1)^2,(-1)^3,...\}$ are solutions to the recurrence, $x_n = x_{n-1} + 2x_{n-2}$

So, choosing any two arbitrary constants C and D, we can verify that

$\{C+D, 2C-D, 2^2D+(-1)^2C, 2^3D+(-1)^3C, ...\}$ is also a solution to the recurrence.

Proof: Suppose $x_n = Ax_{n-1} + Bx_{n-2}$ and $y_n = Ay_{n-1} + By_{n-2}$ for sequences

$\{x_n\}$ and $\{y_n\}$. Then

$Cx_n+Dy_n = C(Ax_{n-1} + Bx_{n-2}) + D(Ay_{n-1} + By_{n-2})$
// by the assumption
// the rest is by algebra
$= CAx_{n-1} + CBx_{n-2} + DAy_{n-1} + DBy_{n-2}$
$= CAx_{n-1} + DAy_{n-1} + CBx_{n-2} + DBy_{n-2}$
$= A(Cx_{n-1} + Dy_{n-1}) + B(Cx_{n-2} + Dy_{n-2})$

The latter term shows that the recursion relation holds for Cx_n+Dy_n

* If you have specific values z_0 and z_1 that you want for the first two terms of the solution sequence, and if you have two sequences $\{x_n\}$ and $\{y_n\}$ that satisfy the recurrence relation, you may be able to solve the two equations $Cx_0+Dy_0 = z_0$ and $Cx_1+Dy_1 = z_1$ for the values of C and D that give z_0 and z_1 as the first two terms of a solution.

* If s and t are two distinct solutions to the characteristic equation, then the sequences $\{1,s,s^2,s^3,s^4,... \}$ and $\{1,t,t^2,t^3,t^4,... \}$ are solutions to the recurrence, and the equations $C+D = z_0$ and $Cs+Dt = z_1$ can be solved to get any two desired numbers z_0 and z_1 for the first two terms of a solution sequence. Since $s \neq t$, the equations are guaranteed to have a solution.

* Since any solution to the recurrence is completely determined by the first two values and the recurrence relation, the information in the previous bullet indicates that all solutions to the recurrence are of the form $\{Cs^n+Dt^n\}$ when s and t are two distinct roots of the characteristic equation.

* Single root case: If the recurrence relation is $x_n = Ax_{n-1} + Bx_{n-2}$ and r is the only root of the characteristic equation $T^2-AT-B=0$, then $(T-r)^2$ is the characteristic equation, which is $T^2-2rT+r^2=0$. Thus $A=2r$, $B=-r^2$ and

$\mathbf{A(k-1)r^{k-1} + B(k-2)r^{k-2}}$
$= Akr^{k-1} + Bkr^{k-2} - Ar^{k-1} - 2Br^{k-2}$
$= Arkr^{k-2} + Bkr^{k-2} - Arr^{k-2} - 2Br^{k-2}$
$= kr^{k-2}(Ar+B) - r^{k-2}(Ar+2B)$

Because r is a root of $T^2-AT-B=0$,

$Ar+B=r^2$

and because $A=2r$ and $B=-r^2$, $Ar = 2r^2 = -2B$, so

Ar+2B=0.

Thus the last quantity in the chain of inequalities above is

$A(k-1)r^{k-1} + B(k-2)r^{k-2}$
$= kr^{k-2}(Ar+B) - r^{k-2}(Ar+2B)$
$= kr^{k-2}(r^2) - r^{k-2}(0)$
$= \mathbf{kr^k}$

This shows that the sequence ***$\{nr^n\}$ is a solution to the recurrence:***

$x_n = Ax_{n-1} + Bx_{n-2}$

(***$\{r^n\}$ is also a solution***, which can be shown in the same way as before.)

* Using the foregoing facts, it can be shown, with a routine proof, that when the characteristic equation has a double root, ***all solutions*** to the recurrence ***are linear combinations of $\{nr^n\}$ and $\{r^n\}$***, and a solution exists for every pair of initial values z_0 and z_1.

Sample illustrative problems for section 5.8

*** Ex 5.8.1: Which of the examples are second-order linear homogeneous recurrence relations with constant coefficients?**

(a) $a_k = 2a_{k-1} - 5a_{k-2}$
(b) $b_k = kb_{k-1} + b_{k-2}$
(c) $c_k = 3c_{k-1}(c^2_{k-2})$
(d) $d_k = 3d_{k-1} + d_{k-2}$
(e) $r_k = r_{k-1} - r_{k-2} - 2$
(f) $s_k = 10s_{k-2}$

*** Ex 5.8.8: (a) find the sequences, based on roots of the characteristic equation, that are solutions to the recurrence relation; and (b) find an explicit formula that satisfies both the initial conditions and the recurrence relation.**

$a_k = 2a_{k-1} + 3a_{k-2}$ for integers $k\geq 2$
$a_0 = 1$; $a_1 = 2$

Day Twenty-Nine: Wednesday, Nov 01, 2017

*** Go over correct way to answer 4.6.12**

Day Thirty: Friday, Nov 03, 2017

*** Ex 5.8.11: Find an explicit formula for the given recurrence relation and initial conditions.**

$d_k = 4d_{k-2}$ for all integers $k\geq 2$
$d_0 = 1$; $d_1 = -1$

*** Ex 5.8.13: Find an explicit formula for the given recurrence relation and initial conditions.**

$r_k = 2r_{k-1} - r_{k-2}$, for all integers $k\geq 2$
$r_0 = 1$ and $r_1 = 4$

Day Thirty-One: Monday, Nov 06, 2017

Topics in Section 5.9 (General Recursive Definitions and Structural Induction)

* Recursive Definition of a Set:

I. BASE: A statement that certain objects belong to the set

Example: "Each symbol of the alphabet is a Boolean expression"

II. RECURSION: A collection of rules indicating how to form new set objects from those already known to be in the set.

Example: "If P and Q are Boolean expressions, then so are (a) $(P \wedge Q)$; (b) $(P \vee Q)$; and (c) ~P."

III. RESTRICTION: A statement that no objects belong to the set other than those coming from I and II.

Example: "There are no Boolean expressions over the alphabet other than those obtained from I and II."

* Definition of a ***string:*** Let S be a non-empty finite set. ***A string over S*** is a finite sequence of elements of S. The elements are called ***characters*** of the string, and the number of characters in the string is the ***length of the string.*** The ***null string*** (aka ***empty string***) is the string with no characters. The null string has **length 0** and is often denoted as **ε** (epsilon).

* ***The Structural Induction form of mathematical induction:***

Let S be a set that has been defined recursively, and consider a proposition (statement) P(x) that may be true or false about objects x in S. To prove that P(x) is true for every x in S:

1. Show that P(x) is true for every object x in the BASE of S.

2. Show that for each rule in the RECURSION, if the rule is applied to an object x in S for which P(x) is true, then P(y) is true for the object y defined by the rule.

Because no objects other than those obtained through the BASE and RECURSION conditions are contained in S, it must be the case that P(x) is true for every object x in S.

Sample illustrative problem for section 5.9

*** 5.9.7: Define S recursively by**

I. BASE: $\varepsilon \in S$

II. RECURSION: if s ∈ S, then bs ∈ S, sb ∈ S, saa ∈ S, and aas ∈ S

III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

Use structural induction to prove that every string in S contains an even number of a's.

Topics in Section 6.1 (Set Theory: Definitions and the Element Method of Proof)

* **The Element Argument: to prove that X ⊆ Y, where X and Y are sets:**

1. Suppose x is a particular but arbitrarily chosen element of X, and
2. Show that x is an element of Y

* **Set Equality:**

  Given sets A and B, A=B if and only if (A ⊆ B and B ⊆ A)

* **Set Operations**

  Suppose A and B are subsets of some 'universal set' U

  The union A ∪ B of A and B is the set of all elements of U that are in either A or B, or both.

  The intersection A ∩ B of A and B is the set of all elements that are in both A and B.

  The difference A-B is the set of all elements of A that are not elements of B.

  The complement A^c of A in U is the difference U-A.

* **Interval Notation for real numbers:**

  Given a,b in R with a≤b

  (a,b) = {x∈R|a<x<b}
  [a,b] = {x∈R|a≤x≤b}
  [a,b) = {x∈R|a≤x<b}
  (a,b] = {x∈R|a<x≤b}
  (a, ∞) = {x∈R|a<x}
  [a, ∞) = {x∈R|a≤x}
  (-∞,b) = {x∈R|x<b}
  (-∞,b] = {x∈R|x≤b}

Day Thirty-Two: Wednesday, Nov 08, 2017

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* **Unions and Intersections of an Indexed Collection of Sets**
* **Alternative Notations: e.g.** $A_0 \cup A_1 \cup \ldots \cup A_n$

• **Definition**

Unions and Intersections of an Indexed Collection of Sets

Given sets A_0, A_1, $A_2, \ldots$ that are subsets of a universal set U and given a nonnegative integer n,

$$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, \ldots, n\}$$

$$\bigcup_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for at least one nonnegative integer } i\}$$

$$\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for all } i = 0, 1, 2, \ldots, n\}$$

$$\bigcap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for all nonnegative integers } i\}.$$

* **The Empty Set.** The empty set is the set with no elements. There is only one empty set. It is denoted $\varnothing$.

* **Disjoint Sets.** Two sets are disjoint if their intersection is the empty set. Example: The intervals (1,2) and (3,4) are disjoint.

* **Pairwise Disjoint Collections of Sets** (aka **Mutually Disjoint**, aka **Non-overlapping**): A collection of sets A_0, A_1, A_2, ... A_n is mutually disjoint if $A_i \cap A_j = \varnothing$ whenever $i \neq j$.

* **A Partition of Sets:** A finite or infinite collection of non-empty sets A_0, A_1, A_2, ... is a partition of a set A if and only if
  1. A is the union of all the A_i.
  2. The sets A_0, A_1, A_2, ... are pairwise disjoint.

* **Power Sets:** The power set $\mathcal{P}(A)$ of the set A is

  the set of all subsets of A.

Day Thirty-Three: Monday, Nov 13, 2017

Quiz #2

Day Thirty-Four: Wednesday, Nov 15, 2017

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* **Cartesian Products**

• Definition

Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements. The **ordered n-tuple, $(x_1, x_2, \ldots, x_n)$,** consists of $x_1, x_2, \ldots, x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. An ordered 2-tuple is called an **ordered pair,** and an ordered 3-tuple is called an **ordered triple.**

Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are **equal** if, and only if, $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$.

Symbolically:

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \quad \Leftrightarrow \quad x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n.$$

In particular,

$$(a, b) = (c, d) \quad \Leftrightarrow \quad a = c \text{ and } b = d.$$

• Definition

Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$ denoted $\boldsymbol{A_1 \times A_2 \times \ldots \times A_n}$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

In particular,

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$$

is the Cartesian product of A_1 and A_2.

Sample illustrative problem for section 6.1
(Definitions and the Element Method of Proof)

* Ex 6.1.4:

A = {n∈Z| n=5k, ∃k∈Z} (integral mults of 5)
B = {n∈Z| n=20h, ∃h∈Z} (integral mults of 20)
Is A ⊆ B? Is B ⊆ A? Explain.

* Ex 6.1.11:

The universal set is the real numbers R. The intervals A=(0,2]; B=[1,4); C=[3,9) are given. Find various unions, intersections, and complements involving A,B,C.

(a) (A ∪ B) = ? ;
(b) (A ∩ B) = ?;
(c) (A^c) = ?
(d) (A ∪ C) = ?;
(e) (A ∩ C) = ?;
(f) (B^c) = ?;
(g) (A^c ∩ B^c) = ?;
(h) (A^c ∪ B^c) = ?;
(i) ($(A \cap B)^c$) = ?;
(j) ($(A \cup B)^c$) = ?;

* Ex 6.1.15: Draw Venn Diagrams to describe sets satisfying certain conditions

(a) (A ∩ B) = ∅, A ⊆ C, (C ∩ B) ≠ ∅
(b) A ⊆ B, C ⊆ B, (A ∩ C) ≠ ∅
(c) (A ∩ B) ≠ ∅, (B ∩ C) ≠ ∅,
(A ∩ C) = ∅, A ⊈ B, C ⊈ B

* 6.1.27 (a) & (d)

(a) Is {{a, d, e}, {b, c}, {d, f}} a partition of {a, b, c, d, e, f}?

(d) Is {{3,7,8}, {2,9}, {1,4,5}} a partition of {1,2,3,4,5,6,7,8,9}?------------------------------------

Day Thirty-Five: Friday, Nov 17, 2017

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* 6.1.31: Find various power sets.
A = {1,2} ; B = {2,3}

Find: $\mathcal{P}(A \cap B)$, $\mathcal{P}(A)$, $\mathcal{P}(A \cup B)$, $\mathcal{P}(A \times B)$,

Topics in Section 6.2 (Properties of Sets)

* Set Identities

Theorem 6.2.1 Some Subset Relations

1. *Inclusion of Intersection:* For all sets A and B,

   (a) $A \cap B \subseteq A$ and (b) $A \cap B \subseteq B$.
2. *Inclusion in Union:* For all sets A and B,

   (a) $A \subseteq A \cup B$ and (b) $B \subseteq A \cup B$.
3. *Transitive Property of Subsets:* For all sets A, B, and C,

   if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Procedural Versions of Set Definitions

Let X and Y be subsets of a universal set U and suppose x and y are elements of U.

1. $x \in X \cup Y \Leftrightarrow x \in X$ or $x \in Y$
2. $x \in X \cap Y \Leftrightarrow x \in X$ and $x \in Y$
3. $x \in X - Y \Leftrightarrow x \in X$ and $x \notin Y$
4. $x \in X^c \Leftrightarrow x \notin X$
5. $(x, y) \in X \times Y \Leftrightarrow x \in X$ and $y \in Y$

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,

   (a) $A \cup B = B \cup A$ and (b) $A \cap B = B \cap A$.
2. *Associative Laws:* For all sets A, B, and C,

   (a) $(A \cup B) \cup C = A \cup (B \cup C)$ and

   (b) $(A \cap B) \cap C = A \cap (B \cap C)$.
3. *Distributive Laws:* For all sets, A, B, and C,

   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and

   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
4. *Identity Laws:* For all sets A,

   (a) $A \cup \emptyset = A$ and (b) $A \cap U = A$.
5. *Complement Laws:*

   (a) $A \cup A^c = U$ and (b) $A \cap A^c = \emptyset$.
6. *Double Complement Law:* For all sets A,

   $(A^c)^c = A$.
7. *Idempotent Laws:* For all sets A,

   (a) $A \cup A = A$ and (b) $A \cap A = A$.
8. *Universal Bound Laws:* For all sets A,

   (a) $A \cup U = U$ and (b) $A \cap \emptyset = \emptyset$.
9. *De Morgan's Laws:* For all sets A and B,

   (a) $(A \cup B)^c = A^c \cap B^c$ and (b) $(A \cap B)^c = A^c \cup B^c$.
10. *Absorption Laws:* For all sets A and B,

    (a) $A \cup (A \cap B) = A$ and (b) $A \cap (A \cup B) = A$.
11. *Complements of U and $\emptyset$:*

    (a) $U^c = \emptyset$ and (b) $\emptyset^c = U$.
12. *Set Difference Law:* For all sets A and B,

    $A - B = A \cap B^c$.

* How to prove two sets A, B are equal:
  Prove (A ⊆ B) and (B ⊆ A)

* Theorem 6.2.4: A set with no elements is a subset of every set.

Proof: By the definition of a subset, if X and Y are two sets,
(X ⊆ Y) iff ∀t, if t∈X then t∈Y
Let S be any set, and suppose that an empty set ∅ is not a subset of S. Then the negation of this statement is true:
(statement 1) ∀t, if t∈∅ then t∈S
That negation is this statement:
(statement 2) ∃t, t∈∅ and t∉S
But statement 2 is false, because there are no elements of ∅, and so there does NOT exist a t such that t∈∅ and t∉S. Therefore statement 2 is false, and its negation, statement 1, is true. In other words, (∅ ⊆ S). Since S was an arbitrary set, this shows that ∅ is a subset of every set.

* There is only one set with no elements, the empty set.

Proof: Suppose that $\varnothing_1$ and $\varnothing_2$ are empty sets. According to the previous proposition,
$\varnothing_1 \subseteq \varnothing_2$ and $\varnothing_2 \subseteq \varnothing_1$. Therefore $\varnothing_1 = \varnothing_2$ by the definition of set equality.

* How to prove a set is empty by contradiction: Assume it has an element and derive a contradiction.

Sample illustrative problem for section 6.2 (Properties of Sets)

* **6.2.3: Fill in the blanks of the following proof that ∀ sets A, B, C, when (A ⊆ B) and (B ⊆ C), (A ⊆ C).**

Proof: Suppose A, B, and C are sets, and that (A⊆B) and (B⊆C). To show that (A⊆C), we must show that every element in ____(a)____ is in ___(b)____. But given any element in A, that element is in ____(c)_____ (because A⊆B), and so that element is also in ____(d)____ (because ____(e)____). Hence (A⊆C).

* **6.2.10: Use an element argument to prove the statement. Assume all sets are subsets of a universal set U.**
  For all sets A, B, and C,
  (A – B) ∩ (C – B) = (A ∩ C) – B

* **6.2.16: Use an element argument to prove the statement. Assume all sets are subsets of a universal set U.**
  For all sets A, B, and C,
  if (A ⊆ B) and (A ⊆ C) then (A ⊆ (B ∩ C))

Day Thirty-Six: Monday, Nov 20, 2017
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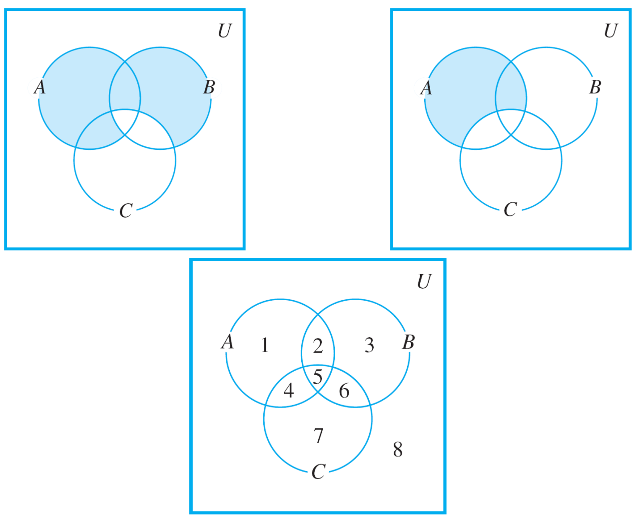
* **Take Roll**

*** 6.2.34: Use the element method for proving a set equals the empty set. (In other words, suppose the set has an element and derive a contradiction.) Assume that all sets are subsets of a universal set U.**

For all sets A, B, and C,
if $(B \cap C) \subseteq A$, then $(C - A) \cap (B - A) = \varnothing$

Topics in Section 6.3 (Disproofs, Algebraic Proofs, and Boolean Algebras)

* A way to construct a counter example to a false set identity



We check the diagram to ascertain the plausibility that

$(A - B) \cup (B - C) = A - C$

is a correct set identity. Using the labels of the sections to construct example sets, we get
$A = \{1,2,4,5\}$, $B = \{2,3,5,6\}$, and $C=\{4,5,6,7\}$.
Here $A-C = \{1,2\}$, $A-B = \{1,4\}$, $B-C = \{2,3\}$, and
$(A - B) \cup (B - C) = \{1,2,3,4\}$

* Proof by induction on $0 \le n \in \mathbb{Z}$ that a set with n elements has 2^n subsets.

Sample illustrative problem for section 6.3 (Disproofs, Algebraic Proofs, and Boolean Algebras)

*** 6.3.1: Find a counter-example: ∀ sets A, B, and C, (all contained in a universal set, U) $(A \cap B) \cup C = A \cap (B \cup C)$**

*** 6.3.17: Prove or give a counter-example: ∀ sets A, B (contained in a universal set U): if $A \subset B$ then $\mathcal{P}(A) \subset \mathcal{P}(B)$.**

*** 6.3.18: Prove or give a counter-example:**

∀ sets A, B (contained in a universal set U):

$\mathcal{P}(A \cup B) \subset \mathcal{P}(A) \cup \mathcal{P}(B)$

Day Thirty-Seven: Wednesday, Nov 22, 2017

*** Take Roll**

*** 6.3.34: Construct an algebraic proof (a proof that uses only the identities of theorem 6.2.2.)**

Prove: ∀ sets A, B, C (contained in a universal set U):

$$(A-B)-C = A-(B \cup C)$$

Topics in Section 8.1 (Relations on Sets)

* Examples of binary relations: $x \le y$ in RxR, and "m-n is even" in ZxZ.
* The inverse of a binary relation: If R is a relation on AxB, then R^{-1} is the relation on BxA given by $(b,a) \in R^{-1}$ if and only if $(a,b) \in R$.
* A relation on a set A is a subset of AxA. It can be represented as a directed graph with a set of nodes representing the elements of A.
* An N-ary relation - a relation on the cross product of N sets - is a subset of that cross product. As objects of logical and mathematical study, relational databases are viewed as N-ary relations.

Sample illustrative problem for section 8.1 (Relations on Sets)

*** 8.1.1: The congruence relation E from Z to Z is defined by**
m E n <--> (m-n) is even, ∀ m,n ∈ Z.

a) Which of these are true? 0E0, 5E2, (6,6)∈E, (-1,7)∈E

b) Prove that nE0 is true for any even integer n.

*** 8.1.9: Let A be the set of all strings of length 4 made of 0's 1's and 2's. Define the relation R on A by: sRt <--> the sum of the characters in s is equal to the sum of the characters in t.**
a) is 0121 R 2200? What about 1011 R 2101, 2212 R 2121, and 1220 R 2111?

11 was a race horse. 22 was 12. When 1111 race, 22112.

*** 8.1.10: Let A = {3,4,5} and B = {4,5,6} and let R be the "less than" relation on AxB. State explicitly which ordered pairs are in R and R^{-1}.**

Day Thirty-Eight: Monday, Nov 27, 2017

*** Take Roll**
*** Hand back Quiz #2 - go over answers**

Day Thirty-Nine: Wednesday, Nov 29, 2017

*** Take Roll**
*** Announce statistics relating to Quiz #2**
*** Info regarding upcoming tests:**
It's likely that quiz #3 and the final will be mixtures of multiple-choice questions and questions calling for the writing of calculations, proofs, or parts of proofs.
*** We did the solution of problem 5.9.6**
*** We went over a sample proof by induction: the sum of the first k cubes is the square of k(k+1)/2**

Day Forty: Friday, Dec 01, 2017

Topics in Section 8.2 (Reflexivity, Symmetry, and Transitivity)

* Definition of reflexive relation on a set, symmetric relation on a set, and transitive relation on a set

Suppose R is a relation on a set A.
R is reflexive if xRx ∀x∈A
R is symmetric if ∀x∈A,y∈A, xRy --> yRx
R is transitive if ∀x∈A,y∈A,z∈A,
[(xRy)∧(yRz)]-->(xRz)

* The relation of "equality" on the set of real numbers is reflexive, symmetric, and transitive. (Any 'reasonably defined' equality relation will be reflexive, symmetric, and transitive. For example equality of sets is reflexive, symmetric, and transitive, and congruence of triangles is reflexive, symmetric, and transitive.)

* The "less than" relation on real numbers is transitive. The "less than or equal" relation on real numbers and the set containment relation are reflexive and transitive, but not symmetric.

* Congruence of integers modulo n is reflexive, symmetric, and transitive.

* Given a relation that is reflexive and symmetric, the relation can be extended to its "transitive closure" -- by adding all the necessary pairs to the relation to make it transitive.

Sample illustrative problem for section 8.2 (Reflexivity, Symmetry, and Transitivity)

8.2.2 A relation is defined on the set A={0,1,2,3}
a. Draw the directed graph.
b. Determine whether the relation is reflexive.
c. Determine whether the relation is symmetric.
d. Determine whether the relation is transitive.
In each case where the relation does NOT have one of the properties, provide a counter example that shows the failure.

R_2 = {(0,0),(0,1),(1,1),(1,2),(2,2),(2,3)}

8.2.4 A relation is defined on the set
A={0,1,2,3}
a. Draw the directed graph.
b. Determine whether the relation is reflexive.
c. Determine whether the relation is symmetric.
d. Determine whether the relation is transitive.
In each case where the relation does NOT have one of the properties, provide a counter example that shows the failure.

R_4 **= {(1,2),(2,1),(1,3),(3,1)}**

8.2.21 Determine whether the given relation is reflexive, symmetric, transitive, or none of these. Justify your answers. Let X = {a, b, c} and $\mathcal{P}(X)$ **the power set of X. A relation L is defined on** $\mathcal{P}(X)$ **as follows: For all A,B∈** $\mathcal{P}(X)$**, A L B ⇔ the number of elements in A is less than the number of elements in B.**

8.2.22 Determine whether the given relation is reflexive, symmetric, transitive, or none of these. Justify your answers. Let X = {a, b, c} and $\mathcal{P}(X)$ **the power set of X. A relation N is defined on** $\mathcal{P}(X)$ **as follows: For all A,B ∈** $\mathcal{P}(X)$**, A N B ⇔ the number of elements in A is NOT EQUAL to the number of elements in B.**

Topics in Section 8.3 (Equivalence Relations)

* **The Relation Induced by a Partition:** Given a partition of a set U (a collection of pairwise disjoint subsets of U whose union is U), we can say x is related to y if x and y are both elements of the same subset of the partition.

* **Theorem 8.3.1:** A relation R induced by a partition of a set U is reflexive, symmetric, and transitive.

* **Definition:** Suppose R is a relation on a set U. R is called *an equivalence relation if it is reflexive, symmetric, and transitive.*

* **Definition:** If R is an equivalence relation on a set U, and if a ∈ U, then *the equivalence class* [a] = {b ∈ U | aRb }.

* **Theorem 8.3.4:** The Partition Induced by an Equivalence Relation: If U is a set and R is an equivalence relation on U, then the distinct equivalence classes of R form a partition of U; that is, the union of the equivalence classes is all of U, and the intersection of any two distinct classes is empty.

* **Definition:** if S is one of the equivalence classes of a relation R on a set U, and if x ∈ S, we say that *x is a representative of S*.

Sample illustrative problem for section 8.3 (Equivalence Relations)

8.3.6 The relation R is an equivalence relation on the set A. Find the distinct equivalence classes of R.

A = {-4, -3, -2, -1, 0, 1, 2, 3, 4, 5}

For all x,y ∈ A, xRy ⇔ 3|(x-y)

Day Forty-One: Monday, Dec 04, 2017

8.3.21
F is the relation defined on Z as follows:
For all m,n ∈ Z, mFn ⇔ 4|(m-n)

(a) Prove that F is an equivalence relation, and
(b) Describe the distinct equivalence classes of F.

Topics in Section 9.4 (The Pigeonhole Principle)

* **The Pigeonhole Principle:** A function from one finite set into a smaller finite set cannot be one-to-one: There must be a least two elements in the domain that have the same image in the co-domain.

(Informally, we say that if n+1 pigeons fly into n holes, then at least two pigeons fly into the same hole.)

The principle is "obvious" but has some application that surprise people. For example, if we choose 5 elements of {1,2,3,4,5,6,7,8} then at least one pair of the elements must add to 9. We can prove this as a consequence of the pigeonhole principle, thinking of mapping each choice to the set among {1,8}, {2,7}, {3,6}, {4,5} containing it.
(5 elements mapping to 4.)

* **The Generalized Pigeonhole Principle:** For any function f from a finite set X with n elements to a finite set Y with m elements, and for any positive integer k, if k < (n/m), then there is some y ∈ Y such that y is the image of at least k+1 distinct elements of X. (There can't be fewer than n/m pigeons in every pigeonhole, because then the total number of pigeons would be less than m*(n/m)=n.)

Sample illustrative problem for section 9.4 (The Pigeonhole Principle)

9.4.4 In a group of 700 people, must there be 2 who have the same first and last initials? Why?

9.4.27 In a group of 2000 people, must at least 5 have the same birthday? Why?

Possible Review Questions:

6.2.26: Use the element method to prove that for all sets A, B, C of some universal set,

$(A-C) \cap (B-C) \cap (A-B) = \varnothing$

In other words, assume the set has an element and derive a contradiction.

6.2.19: Prove for all sets A, B, C of some universal set, $A \times (B \cap C) = (A \times B) \cap (A \times C)$

5.8.9: Suppose that $t \neq 0$ is a number and that this sequence

$b_0=1=t^0$, $b_1=t=t^1$, $b_2=t^2$, $b_3=t^3$, ... $b_n=t^n$ ...

satisfies this recurrence relation:

$b_k = 7b_{k-1} - 10b_{k-2}$ for all integers $k \geq 2$

(a) Find all the values of t that "work" and do a check to makes sure your answers are correct.

Day Forty-Two: Wednesday, Dec 06, 2017

(b) Suppose the recurrence above is given and also the "initial conditions" $b_0=2=b_1$. Find an an explicit formula for the solution to the recurrence and initial conditions.

5.8.15: Same kind of problem as 5.8.9, except the recurrence is

$h_k = 6h_{k-1} - 9h_{k-2}$ for all integers $h \geq 2$ and the initial conditions are $h_0=1$ and $h_1=3$.