Math 2300, Section #1; 14:00-14:50 MWR P-102; Daily Notes Fall, 2016

Day One: Wed, Aug 24, 2016

----

* **Take Roll**
* Tell students that they are required to carefully read the entire course description on line
  (or request a printed full copy if not able to read on line)

* **Go over items in course description**
  + Name
  + Office & Hours
  + E-mail
  + Homepage + Class web site
* Go over Course Objectives - what do folks think those objectives mean?
* **Text**
  + Problems getting one?
  + How many will be using Ed #4? How many using Ed #3?
  + Do students know about the book web site - go and check out the errata and review materials with the class

* **Course components**
  + HW (25%)
  + 2 or more quizzes (50%)
  + comprehensive final exam (25%)

  Exception: You fail the course if you get a failing average on tests. Otherwise I use the weights above.

HOMEWORK
Look at the Assignments with Class -- First Assignment is TODAY

Make sure to be familiar with homework submission rules in the course description -- **notice strict late policy** -- anybody here not able to access the course description, ask me for a paper copy.

Schedule - look at the schedule with the class and go over this week's assignments -- There is reading and homework. The first HW is due next Wednesday.

Lecture
Chapter One - Chapter One is about language we use to make it easier to talk about mathematical concepts, and to make it easier to find solutions to math problems.

Section 1.1 - Variables
Section 1.2 - The Language of Sets
Section 1.3 - The Language of Relations and Functions

Section 1.1 - Variables are names we use for numbers or other things.

Example: (x)
We can use a variable as a names of a specific numbers that we don't know.
Example: x = 23981 times 53204

When we want to state a fact that is true about a lot of numbers, we can use a variable as a name that represents any one of those numbers.
Example: 2(y + 1) - y = y + 2

If we are wondering if a number with a certain property exists, we can use a variable to represent the number, to help us find the number, or to prove it does NOT exist.
Example: Can 6 be the sum of a number and its square?
With variable: x^2 + x = 6. Using quadratic formula, we find that 2 and -3 work.

It's handy to use variables as names for numbers if we write some mathematical information that is long. We can use a variable name to refer to something over and over again in different parts of what we write. The reader understands what we mean, because we use the variable name consistently.

People often use variables to make mathematical statements
Universal: All real numbers y have the property that 2(y + 1) - y = y + 2
Conditional: If x=2 or x=-3 then x^2 + x = 6
Existential: There are two numbers x,y such that x^2 = 25, and y^2 = 25.

Day Two: Thurs, Aug 25, 2016
----

* Take Roll
* Take care of remaining problems with adding the class
* Today's HW assignment is problems for section 1.2 on page 13: 2,4,7,9,12; Due on Thursday, September 1.
* Wednesday's assignment is 1.1 problems on page 5, 2,4,7,11,13; due Wednesday, Aug 31.
* The goal today is to come close to finishing discussion of chapter one.

Ask students for questions about the assigned HW.

Work problems with the class: 1.1.3, 1.1.10

Section 1.2 - The Language of sets

Sets are basically just collections of things - kind of like a basket, containing some things, or maybe empty.

The things that are in the set are called elements of the set.

If S is a set and x is an element of S, we use the notation $x \in S$ to denote that.

We can indicate the elements of a set by using "roster" notation with or without ellipsis:

S={a, t, r} ;
T= {1, 2, ..., 14};
V = {31.2, 41.2, 51.2, ... }

When you use ellipsis you need to be sure that the people who read your writings will understand what you mean by the ellipsis. That's why something is written in the beginning to establish the pattern.

The idea of a set does not include any notion of the order of the elements.
S={1,2,3}; T={2,1,3}; and U={2,3,1} are all the same set, because the elements are the same.

Also, an element is either in the set or not. An element can't be in a set twice.
For example {1,2,2} = {1,2}

Familiar sets R, Z, Q, R+, Z-, etc

R and Z illustrate the difference between the idea of continuous and discrete mathematics

Tell students to work as many HW problems as possible between now and Monday, and come in with their questions about how to work them.

Show class some example of "**set builder**" **notation** - say using inequality conditions

There's roster notation to specify a set - like **S = {-1, 0, 1, 2, 3}.**
You can also use "SET BUILDER" notation to specify a set:

For example the set S above is also described as

$$S = \{ n \in Z \mid -1 \le n \le 3 \}$$

It's the same set - You read the notation this way:

"The set of all n in Z (integers) such that n is between -1 and 3 (inclusive).

The **set builder notation is generally** **<u>more expressive</u>** than the roster notation. In other words, it's often easier to say what you mean using the set builder notation. For **example:**

C1 = { m ∈ Z | m = pq, where p and q are primes}

Imagine trying to use roster notation to specify the set C1.

Notions of **subset, proper subset, equal sets, difference between** being a **subset and** being an **element.**

Subset: if A and B are sets, then
A ⊆ B means that **every element of A is also an element of B**

Example:
{2, 3, 1} ⊆ {-20, 2.3, 1, 3, 2, 4}
Example: Z ⊆ R
Example: {1,2} ⊆ {1,2}

Proper Subset: if A is a subset of B, and if **there is at least one element of B that is NOT in A**, then A is a proper subset of B.

Example of Proper Subset Relation: {2, 3, 1} ⊆ {-20, 2.3, 1, 3, 2, 4}
Example of Proper Subset Relation:
Z ⊆ R

Equality of Sets: If A and B are sets such that **A ⊆ B AND B ⊆ A**, then A and B are equal sets. In that case every element of A is an element of B, and every element of B is an element of A.

Example:
{7, 7, 2, 3, 1} = {1, 1, 2, 3, 7}

Difference between being an element and being a subset.
⊆ and ∈ mean completely different things.
For example {2, 5} ⊆ {5, 2, 4} but {2, 5} is not an element of
{5, 2, 4}. The only elements of
{5, 2, 4} are integers, and {2, 5} is not an integer.
Another example: a singleton set is not the same as its element: 4 and {4} are not the same thing.
4 ∈ {4}, but neither is a subset of the other.
The empty set is the set with no elements. ∅

The empty set is a subset of every set. This fact is **"vacuously" true.**
If S is a set, then ∅ ⊆ S is true. **Every element of ∅ is an element of S.**
That's a vacuously true statement, because the empty set has no elements.

A slightly different way of looking at this: If ∅ is **NOT** a subset of every set, then **there must be some set K such that ∅ is NOT a subset of K.** For

THAT to be true, **there must be an element of ∅ that is not an element of K.** But since there are NO elements of ∅, there isn't one that's not in K.

Idea of Cartesian product, how to express ordered pairs as sets, and as "tuples"

Named after mathematician Rene Descartes, the Cartesian product of a set A and a set B is

A x B = { (a,b) | a ∈ A and b ∈ B }

(a,b) is called an ordered pair. It is not the same as the set {a,b} because the it has **a first element a, and a second element b.** So although {a,b} = {b,a},

it is not the case in general that **(a,b) = (b,a).** (a,b) = (b,a) only when a = b.

When mathematicians need **to be very precise** about what they mean by an ordered pair, they can use set notation, and this definition:

(a,b) is defined as { {a}, {a,b} }

When a ≠ b, **a is distinguished** as the 'first' element of the ordered pair **by the fact that it is in both of these sets: {a} {a,b}.**

(a,b) = (c,d) iff a=c and b=d.

Day Three: Mon, Aug 29, 2016
----

* Take Roll

Note: We kind of started here on Monday, but I had not lectured about some of the latter set-related definitions above yet, so I used the time working on the problems to introduce (most of) the remaining concepts.

Work problems with the class:
1.2.3:
a. Is 4 = {4}?
b. How many elements are in the set {3,4,3,5}?
c. How many elements are in the set {1, {1}, {1, {1}}}?

1.2.5
Which of the following sets are equal
A={0,1,2}
B={x∈R|-1≤x<3}
C={x∈R|-1<x<3}

Math 2300, Section #1; 14:00-14:50 MWR P-102; Daily Notes Fall, 2016

$D=\{x\in Z \mid -1<x<3\}$
$E=\{x\in Z^{+} \mid -1<x<3\}$

1.2.8:
$A = \{c, d, f, g\}$
$B = \{f, j\}$
$C = \{d, g\}$
Answer each question, give reasons.
a. Is $B \subseteq A$?
b. Is $C \subseteq A$?
c. Is $C \subseteq C$?
d. Is C a proper subset of A?

1.2.10:
a. Is $((-2)^2, -2^2) = (-2^2, (-2)^2)$?
b. Is $(5,-5)=(-5,5)$?
c. Is $(8-9,(-1)^{1/3}) = (-1,-1)$?
d. Is $(-2/(-4), (-2)^3) = (3/6, -8)$?

1.2.11:
$A = \{w, x, y, z\}$
$B = \{a,b\}$

Use the set-roster notation to write each of the following sets, and indicate the number of elements that are in each set:

a. A x B
b. B x A
c. A x A
d. B x B

(1.2.11)Set-roster notation
(1.2.3b,c)Sets are not multi-sets
(1.2.5)R, Z, Q, etc
(1.2.5)Set-builder notation
(1.2.8)Subsets
(1.2.8)Proper Subsets
(1.2.5)Equality of sets
(1.2.3a,c) Distinction between element and subset relations
(1.2.11)Cartesian products
ordered pairs
(1.2.10a,b,c)equality of ordered pairs
Set notation for ordered pair
==================================

Day Four: Wednesday, Aug 31, 2016
----

* **Take Roll**

Class work on SECTION 1.3, the LANGUAGE OF RELATIONS AND FUNCTIONS.

* (3.1,3.5) **Defining a relation** between elements of A and elements of B as a subset of the cartesian product A x B.
* (3.1c) **Domain** of a relation (A in A x B)
* (3.1c,3.13) **Co-domain** of a relation (B in A x B)
* (3.1d,3.13) **Arrow diagram** of a relation & **Graph** of a relation
  + **One-one** if at most one arrow into each item in the Co-domain
  + **Onto** if at least one arrow into each item in the Co-domain
  + **Function** if there is one unique arrow from each item in the Domain to an element in the CoDomain.
* (3.5,3.11) Graph of a relation
* (3.11,3.13) **Functions**
* (3.11,3.13) **Functions** as types of relations (**exactly one arrow from each domain element**)
* (3.13) x, **F(x) notation** for function F
* (3.11) **vertical line test** to determine if a 'graph' is a function
* **Functions as machines** - black boxes
* **Equality of functions** - point-wise

Sample problems to work (pp 21-26):

1.3.1:
A={2,3,4}
B={6,8,10}
Relation R defined by
(x,y) element of AxB, y/x is an integer.
a. is 4R6?, Is 4R8?, Is (3,8) an element of R? Is (2,10) an element of R?

b. Write R as a set of ordered pairs
c. Write the domain and co-domain of R.
d. Draw an arrow diagram for R.

1.3.5:
Define a relation S from R to R as follows: For all (x,y) in RxR,
(x,y) belongs to R means x ≥ y.

a. Is (2,1) in S? Is (2,2) in S?
Is 2S3? Is (-1)S(-2)?
b. Draw the graph of S in the Cartesian plane.

1.3.11: Define a relation P from R^+ to R as follows:

For all real x,y with x>0,
(x,y) are related if $x=y^2$.

Is P a function? Explain.
If P is a function, for each x in R^+, there must be a unique y in R such that $x=y^2$

1.3.13
Let A={-1,0,1} and B={t,u,v,w}. Define a function F:A--->B with a diagram.

-1	**t**
0	**u**
1	**v**
	w

(diagram indicates F(-1)=F(1)=u, and F(0)=w)

a. Write the domain and codomain of F.
b. Find F(-1), F(0), and F(1).

Day Five: Thursday, Sep 01, 2016

I think we started with problem 3.13 that day, and worked through to almost finish problem 2.1.6.

Finish up with problems 3.5, 3.11, and 3.13 from last week.

Topics in Section 2.1
+ (3)(Forms of) Logical arguments
+ (J1) Statements (i.e. propositions)
+ (6) Using ~, ∨, ∧ to make compound statements
+ (32) Expressing inequalities using and, or
+ (16,18) truth values
+ (6) conjunction $p \wedge q$
+ (6) disjunction $p \vee q$
+ Evaluating compound statements in general
+ Exclusive OR
+ (16,18) Logical Equivalence - same truth values for all substitutions of values for variables.
+ (30,32) Negations of conjunctions and disjunctions - De Morgan's Laws
+ (40,41) Tautologies - always true no matter the value of variables
+ (40,41) Contradictions - always false no matter the value of variables

+ (48) Laws of Boolean algebra (e.g. commutative, associative, ...)
+ (48) simplifying statements using Boolean algebra

Problems to work

2.1.3

J1: Which of the following is a statement?
(a) That's what I like about you.
(b) Tupelo is a kind of tree.

Day Six: Wednesday, Sep 07, 2016
----

* **Take Roll**

With problems, start with 2.1.6(b), but keep in mind that I haven't yet done much discussion of the concepts in section 2.1.

2.1.6,
Write the sentences in symbolic form Using ~, ∨, ∧ to make compound statements
s = "stocks are increasing"
i = "interest rates are steady"

a. Stocks are increasing but interest rates are steady
b. Neither are stocks increasing, nor are interest rates steady.

2.1.16, 2.1.18,
Use a truth table to determine if the two forms are logically equivalent - include a sentence justifying your answer, and showing you understand the meaning of logical equivalence. ~, ∨, ∧

16: $p \vee (p \wedge q)$ versus p
18: $p \vee t$ versus t

maybe do $\sim(p \wedge q)$ versus $\sim p \wedge \sim q$,
to illustrate non-equivalence - this is an example from section 2.1.

2.1.30, 2.1.32,
Use one of DeMorgan's Laws to write a logical negation:

30: "The dollar is at an all-time high AND the stock market is at a record low."

Negation: The dollar is NOT at an all-time high OR the stock market is NOT at a record low.

Day Seven: Thursday, Sep 08, 2016

----

* **Take Roll**

32: $-2 < x < 7$

x <= -2 OR x >= 7
(Show graph for this.)

Make a truth table to prove one of DeMorgan's laws:

$\sim(p \wedge q) \iff (\sim p) \vee (\sim q)$
$\sim(p \vee q) \iff (\sim p) \wedge (\sim q)$

2.1.40, 2.1.41,
Use truth tables to figure out which are tautologies and which are contradictions:

40: $(p \wedge q) \vee (\sim p \vee (p \wedge \sim q))$
41: $(p \wedge \sim q) \wedge (\sim p \vee q)$

Day Eight: Monday, Sep 12, 2016

----

* **Take Roll**

Begin by finishing this up:
Use truth tables to figure out which are tautologies and which are contradictions:

41: $(p \wedge \sim q) \wedge (\sim p \vee q)$

2.1.48 A logical equivalence is derived below. Supply a reason for each step. (See p. 35 of Epp for the list of equivalences.)

$(p \wedge \sim q) \vee (p \wedge q) \equiv p \wedge (\sim q \vee q)$ by <u>(a)</u>

$\equiv p \wedge (q \vee \sim q)$ by <u>(b)</u>

$\equiv p \wedge \mathbf{t}$ by <u>(c)</u>

$\equiv p$ by <u>(d)</u>

<u>(a) is distributive law (3)</u>
<u>(b) is commutative law (1)</u>
<u>(c) is negation law (5)</u>
<u>(d) is identity law (4)</u>

Topics in Section 2.2 (Conditional Statements)

* Hypotheses (antecedents), conclusions (consequents), and conditional statements
* Precedence of --> among ∨, ∧ , and ~ (--->last, ~ first)
* Truth tables for conditional statements
* Representation of if-then as OR [(p-->q) ≡ (~p ∨ q)]

MAKE SURE TO USE THE ABOVE AS THE DEFINITION OF (p-->q)

* Negation of conditional statement (~(~p ∨ q) ≡ (p ∧ ~q))
* Equivalence of a conditional statement and its contrapositive ((p-->q) ≡ (~q --> ~p))

  Maybe the easiest thing is to start with examples of contrapositives.

* Converse and inverse of a conditional statement [Conditional (p-->q)], [Inverse of the Conditional: (~p-->~q)], [Converse of the Conditional (q-->p)]
* Only if and the bi-conditional. In logic the phrase p only if q is logically equivalent to p --> q
* If and only if p <--> q is logically equivalent to (p-->q) ∧ (q-->p)
* Necessary and sufficient conditions

  "r is sufficient for s" means r-->s

  "r is necessary for s" means (~r --> ~s) ≡ (s-->r)

  "r is necessary and sufficient for s" means (s<-->r), which is the same meaning as s if and only if r.

Sample illustrative problems:

2.2.2 Re-write in if-then form: "I am on time for work if I catch the 8:05 bus"

Day Nine: Wednesday, Sep 14, 2016

----

* **Take Roll**

Rewrite in if-then form:

J2: Bear to the right or you'll get into a collision

2.2.16 Write the two statements in symbolic form and determine whether they are logically equivalent. Include a truth table and a few words of explanation.

* If you paid full price, you didn't buy it at Crown Books.

* You didn't buy it at Crown Books or you paid full price.

2.2.19 Write the negation of "If Sue is Luiz' mother, then Ali is his cousin."

2.2.24 Use truth tables to establish the truth of this statement:

"A conditional statement is NOT logically equivalent to its converse."

2.2.32 Rewrite as a conjunction of two if-then statements:
"This quadratic equation has two distinct real roots if, and only if, its discriminant is greater than zero"

2.2.34 Rewrite the statement in if-then form in two ways, one of which is the contrapositive of the other.
"The Cubs will win the pennant only if they win tomorrow's game."

2.2.40 Rewrite in if-then form: "Catching the 8:05 bus is a sufficient condition for my being on time for work."

Topics in Section 2.3 (Valid and Invalid Arguments)

* An argument is a series of statements - the last statement is called the conclusion, and the others are called premises
* Valid argument - conclusion must be true if premises are true
  in other words it is impossible for the conclusion to be false when the premises are true.
* Inferred, deduced
* Testing an argument for validity with a truth table
* Critical row of a truth table
* Syllogisms(two premises + conclusion), major premise, minor premise
* Modus Ponens (p-->q; p; therefore q)
* and Modus Tollens (p-->q; ~q; therefore ~p)
* Rules of inference
* Generalization (p ∴ p ∨ q; q ∴ p ∨ q)
* Specialization: (p ∧ q, ∴ p; p ∧ q, ∴ q)
* Elimination (p ∨ q, ~q, ∴ p; p ∨ q, ~p, ∴ q)
* Transitivity (p-->q, q-->r, ∴ p-->r)
* Proof by division into cases (p ∨ q, p-->r, q-->r, ∴ r)
* Fallacies
* The converse error (the fallacy of affirming the consequent)
* The inverse error (the fallacy of denying the antecedent)
* Sound argument - valid and all premises true
* Unsound argument - any argument that is not sound
* Contradiction Rule (~p-->**c**, [c is a contradiction] ∴ p)

Sample illustrative problems:

2.3.8: Use truth tables to determine whether (p ∨ q, p-->~q, p-->r, ∴ r) is valid. Indicate which columns represent premises, and which represents the conclusion. Explain how the truth table supports your answer. Show you understand the definitions of valid/invalid.

Day Eleven: Monday, Sep 19, 2016
----

* **Take Roll**
* **Last Friday, I assigned a HW due this coming Thursday.**

More sample illustrative problems from section 2.3:

2.3.22: Use symbols to write the logical form of the argument. Then use a truth table to test the argument for validity. Indicate which columns indicate the premises and which represents the conclusion. Include words of explanation showing you understand the meaning of validity.

If Tom is not on Team A, then Hua is on Team B.
If Hua is not on Team B, then Tom is on Team A.
∴ Tom is not on Team A, or Hua is not on Team B.

2.3.27: Write the logical form of the argument with symbols. If valid, identify the rule of inference that assures its validity. Otherwise state whether the inverse or converse error is made.

If this number is larger than 2, then its square is larger than 4.
This number is not larger than 2.
∴ The square of this number is not larger than 4.

Day Twelve: Wednesday, Sep 21, 2016

(There will probably be several "date boundaries" missing from here until October.)

Topics in Section 3.1 (Predicates and Quantified Statements I)

* **Predicate calculus** (the symbolic analysis of predicates and quantified statements)
* **Statement calculus** (also: propositional calculus - the symbolic analysis of ordinary compound statements)
* **Predicate symbol** - obtained by removing some or all nouns from a statement (for example: "is a student at Bedford College", or "is a student at")
* **Predicate variables** - "when concrete values are substituted in place of predicate variables, a statement results." -- for example x and y are the predicate variables here: "x is a student at y".

Math 2300, Section #1; 14:00-14:50 MWR P-102; Daily Notes Fall, 2016

* **Predicate** (also: propositional function or open sentence - a predicate symbol together with suitable predicate variables)
* **Domain of a predicate variable** (the set of all values that may be substituted in place of the variable)
* **Truth set of a predicate P(x)** (the set of all values x in the domain of the variable that make P(x) true - $\{x \in D \mid P(x)\}$)
* **The Universal Quantifier** ∀ (for all, for each)
* **Truth or Falsity of a Universal Statement** (true if and only if substitution of every element of the domain of the variable(s) yields a true statement)
* **The Existential Quantifier** ∃ (there exists, for some)
* **Truth or Falsity of an Existential Statement** (true if and only if the substitution of at least one value in the domain make a true statement)
* **Universal Conditional Statements** (for example: $\forall x, P(x) \to Q(x)$)
* **Equivalent Forms of Universal and Existential Statements**
* **Implicit Quantification** (for example: "if $n \in Z$ then $n \in Q$", or "24 can be written as the sum of two even integers") The first actually contains a universal quantification, and the second contains an existential quantification.

Sample illustrative problems:

3.1.5: Let Q(x,y) be the predicate "If $x<y$ then $x^2 < y^2$" with domain for both x and y being the set **R** of real numbers.

a. Explain why Q(x,y) is false if x = -2 and y=1.

b. Give values different from those in part (a) for which Q(x,y) is false.

c. Explain why Q(x,y) is true if x=3, and y=8.

d. Give values different from those in part (c) for which Q(x,y) is true.

3.1.13: Consider the following statement:

∀ basketball players x, x is tall.

Which of the following are equivalent ways of expressing this statement?

a. Every basketball player is tall.
b. Among all the basketball players, some are tall.
c. Some of all the tall people are basketball players.
d. Anyone who is tall is a basketball player.
e. All people who are basketball players are tall.
f. Anyone who is a basketball player is a tall person.

3.1.16a: Rewrite in the form "∀ ________ x, _________"
"All dinosaurs are extinct."

Math 2300, Section #1; 14:00-14:50 MWR P-102; Daily Notes Fall, 2016

3.1.16c: Rewrite in the form "$\forall$ ________ x, _________"
"No irrational numbers are integers."

3.1.16e: Rewrite in the form "$\forall$ ________ x, _________"
"The number 2,147,581,953 is not equal to the square of any integer."

3.1.32: Let R be the domain of the predicate variable x. Which of the following are true and which are false? Give counter examples for the statements that are false.

(a) $(x>0) \Rightarrow (x>1)$

(c) $(x^2 > 4) \Rightarrow (x > 2)$

Topics in Section 3.2 (Predicates and Quantified Statements II)

* **Negating Quantified Statements:** $(\sim(\forall\ x\in D,\ P(x)) \equiv (\exists\ x\in D,\ \sim P(x)))$
* **Relation among $\forall$, $\exists$, $\vee$, $\wedge$**
  When $D = \{x_1, x_2, \ldots, x_n\}$ $(\forall\ x\in D,\ P(x)) \equiv (P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n))$
  and $(\exists\ x\in D,\ P(x)) \equiv (P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n))$

* **Vacuous truth of a universal statement** "For all balls on the table x, if x is in this (empty) bowl, x is blue"
* **Universal conditional statements**
* **Meaning of *necessary*, *sufficient*, and *only if*** in relation to quantified statements

$(\forall\ x\in D$, $P(x)$ is sufficient for $Q(x)) \equiv (\forall\ x\in D,\ P(x) \dashrightarrow Q(x))$
$(\forall\ x\in D$, $P(x)$ is necessary for $Q(x)) \equiv (\forall\ x\in D,\ Q(x) \dashrightarrow P(x))$
$(\forall\ x\in D$, $P(x)$ only if $Q(x)) \equiv (\forall\ x\in D,\ P(x) \dashrightarrow Q(x))$

* **Logical equivalence of quantified statements:** identical truth values, no matter what predicates are substituted for the predicate symbols, and no matter what sets are used for the domains of the predicate values. For example both $\sim(\forall\ x\in D,\ P(x))$ and $(\exists\ x\in D,\ \sim P(x))$ have the same truth values, no matter what predicate P is, or what set D is.
* **Negation of an Existential Statement:** $\sim(\exists\ x\in D,\ P(x)) \equiv (\forall\ x\in D,\ \sim P(x))$
* **Negation of a Universal Conditional:**
  $\sim(\forall\ x\in D,\ P(x) \to Q(x))) \equiv (\exists\ x\in D,\ P(x) \wedge \sim Q(x))$

* **Variants of Universal Conditional Statements**

(i) Statement: $(\ \forall\ x\in D,\ P(x) \to Q(x)\ )$
(ii) Contrapositive: $(\ \forall\ x\in D,\ \sim Q(x) \to \sim P(x)\ )$
(This is logically equivalent to i)
(iii) Converse: $(\ \forall\ x\in D,\ Q(x) \to P(x)\ )$
(This is NOT logically equivalent to i)
(iv) Inverse: $(\ \forall\ x\in D,\ \sim P(x) \to \sim Q(x)\ )$

(This is logically equivalent to iii
- it is the contrapositive to iii)

Sample illustrative problems for section 3.2

3.2.3a: Write a formal negation for (∀ fish x, x has gills.)
3.2.3c: Write a formal negation for
(∃ a movie m such that m is over 6 hours long.)

3.2.16: Write a negation for (∀ real numbers x, if x2 >= 1 then x >0.)
3.2.22: Write a negation for (If the square of an integer is odd, then the integer is odd.)
3.2.32: Write the converse, inverse, and contrapositive of this statement:
(If the square of an integer is odd, then the integer is odd.)
Indicate which among the four statements is true, and which is false.
Give counter examples for the ones that are false.

Topics in Section 3.3 (Statements with Multiple Quantifiers)

* Truth of a ∀ ∃ Statement (for example, in a "Tarski World")
* Truth of a ∃ ∀ Statement (for example, in a "Tarski World")
* Key idea for the two above - imagine making "choices" in the order the quantifiers are given.
* Interpreting Multiply-Quantified Statements

Translating from Informal to Formal Language

Negations of Multiply-Quantified Statements:

Example #1

$\sim(\forall\ x\in D,\ \exists\ y\in E \text{ such that } P(x,y))$
$\equiv (\exists\ x\in D,\ \sim(\exists\ y\in E \text{ such that } P(x,y)))$
$\equiv (\exists\ x\in D,\ \forall\ y\in E,\ \sim P(x,y))$

Example #2

$\sim(\exists\ x\in D,\ \forall\ y\in E,\ P(x,y))$
$\equiv (\forall\ x\in D,\ \sim(\forall\ y\in E\ P(x,y)))$
$\equiv (\forall\ x\in D,\ \exists\ y\in E,\ \sim P(x,y))$

* Order of Quantifiers
  - If you interchange ∀ and ∃, usually it changes the meaning
    Example: ∀ people x, ∃ a person y such that x loves y
* Formal Logical Notation

Go over the Tarski World Examples 3.3.1, 3.3.2, 3.3.9
Do a&b of Example 3.3.10: Formalizing Statements in a Tarski World

Sample illustrative problems for section 3.3

3.3.9a: Let D = E = {-2, -1, 0, 1, 2} Explain why the following statement is true: $\forall\ x\in D,\ \exists\ y\in E$ such that $x+y=0$.

3.3.15: (a) Rewrite the statement in English, without using the symbols ∀ or ∃ or variables and expressing your answer as simply as possible, and (b) write a negation for the statement: ∀ odd integers n, ∃ an integer k such that n=2k+1.

3.3.34 (a) Rewrite the statement formally using quantifiers and variables, and (b) write a negation for the statement: Somebody loves everybody

3.3.37 (a) Rewrite the statement formally using quantifiers and variables, and (b) write a negation for the statement: Any even integer equals twice some integer.

Topics in Section 3.4 (Arguments with Quantified Statements)

* The rule of universal instantiation: If some property is true of everything in a set, it is true of any particular thing in the set.

* Universal Modus Ponens:
  ∀ x, If P(x) then Q(x);
  P(a) for a particular a,
  ∴ Q(a)

* Universal Modus Tollens:
  ∀ x, If P(x) then Q(x);
  ~Q(a) for a particular a,
  ∴ ~P(a)

* Validity of Arguments with Quantified Statements
* The Quantified Form of the Converse Error
  ∀ x, If P(x) then Q(x);
  Q(a) for a particular a,
  ∴ P(a) <--- invalid conclusion

* The Quantified Form of the Inverse Error
  ∀ x, If P(x) then Q(x);
  ~P(a) for a particular a,
  ∴ ~Q(a) <--- invalid conclusion

* Universal Transitivity
  ∀ x, P(x) --> Q(x)
  ∀ x, Q(x) --> R(x)
  ∴ ∀ x, P(x) --> R(x)

Sample illustrative problems for section 3.4

3.4.7: The argument may be valid by universal modus ponens or universal modus tollens. It may be invalid and exhibit the converse or inverse error. State whether valid or invalid and justify your answer:

All healthy people eat an apple a day
Keisha eats an apple a day
∴ Keisha is a healthy person

3.4.8: The argument may be valid by universal modus ponens or universal modus tollens. It may be invalid and exhibit the converse or inverse error. State whether valid or invalid and justify your answer:

All freshmen must take writing
Caroline is a freshman
∴ Caroline must take writing

* 3.4.23: Indicate whether the argument is valid or invalid. Support your answer by drawing diagrams.

All teachers occasionally make mistakes.
No gods ever make mistakes.
∴ No teachers are gods.

* 3.4.24: Indicate whether the argument is valid or invalid. Support your answer by drawing diagrams.

No vegetarians eat meat.
All vegans are vegetarian.
∴ No vegans eat meat.

Chapter Four (Elementary Number Theory and Methods of Proof)

Topics in Section 4.1 (Proof and Counterexample I: Introduction)

* Definitions of even and odd integers (2k or 2k+1)
* Definition of a prime number: (p>1 in Z st if n,m in Z^+ and nm=p, then n=p or m=p)
* Definition of a composite number: (c>1 in Z, and c=nm for integers n>1 and m>1)
* Constructive Proofs of Existential Statements (to show ∃ x∈D s.t. Q(x), one can either find an x that makes Q(x) true, or give a set of directions for finding an x that makes Q(x) true. Either way, that's a constructive proof.)
* Non-Constructive Proof of Existence: (a) show existence is guaranteed by some theorem, or (b) show that the assumption that there is no x that makes Q(x) true leads to a contradiction.

* Disproving Universal Statements by Counterexample (Show P(x) ---> Q(x) is false by providing a counter example c such that P(c) && ~Q(c))
* The Method of Exhaustion (show P(x) ---> Q(x) by individually checking each x such that P(x) is true.)

* Generalizing from the Generic Particular (e.g. "Direct Proof": show P(x) ---> Q(x) by assuming x is some generic element of the domain, and basing the demonstration only on that)
* Existential Instantiation - if you have established that something exists, you can give it a name in your logical arguments, so long as you don't give it a name that is already being used for something else. (Example: if we know that m is an even number, then we know it is twice some integer, so we can give that integer the name k, and write m = 2k.)

Sample illustrative problems for section 4.1

8: Prove there is a real number x>1 such that $2^x > x^{10}$.

The Homework problem (#10) is to prove that there is an integer n such that $2n^2 - 5n + 2$ is prime. Hint: $2n^2 - 5n + 2$ can be factored as (n-2)*(2n-1)

Prove the statement. Use only the definitions of the terms and the Assumptions listed on page 146, not any previously established properties of odd and even integers. Follow the directions in this section for writing proofs of universal statements.

30: For all integers m, if m is even then 3m+5 is odd.

35: Prove the statement is FALSE: There exists an integer m >= 3 such that (m^2-1) is prime.

39: Find the mistake in the "proof":
Theorem: The difference between any odd integer and any even integer is odd.
"Proof: Suppose n is any odd integer, and m is any even integer. By definition of odd, n=2k+1, where k is an integer, and by definition of even, m=2k, where k is an integer. Then n-m = (2k+1)-2k = 1. But 1 is odd. Therefore, the difference between any odd integer and any even integer is odd."

55: Determine whether the statement is true or false. Justify your answer with a proof or counter-example, as appropriate. Use only the definitions of the terms and Assumptions listed on page 146, not any previously established properties.

Every positive integer can be expressed as a sum of three or fewer perfect squares.

50: Determine whether the statement is true or false. Justify your answer with a proof or counter-example, as appropriate. Use only the definitions of the terms and Assumptions listed on page 146, not any previously established properties.

For all integers n and m, if n-m is even then n^3-m^3 is even.

Topics in Section 4.2 (Proof and Counterexample II: Rational Numbers)

* Rational Numbers (Q): Q = { r∈R | r = a/b, for integers a,b, with b≠0}
  They are called rational because they are ratios of integers

* More on Generalizing from the Generic Particular - seen as being prepared to meet the challenge of showing P(x) is true for any particular value of x that is offered by an 'adversary'.

* Deriving New Mathematics from Old - Once you have proved something, you can use it in proofs of new things.

* A Corollary is a statement whose truth can be immediately deduced from a theorem that has already been proved.

Sample illustrative problems for section 4.2

* H16: The quotient of any two rational numbers is rational.
* 39: Find the flaw in the proof that the sum of two rational numbers is rational:
  "PROOF: Suppose r and s are rational numbers. If r+s is rational then by definition of rational r+s = a/b where a,b are integers and b≠0. Also, since r and s are rational, r=i/j, s=m/n for integers i,j,m,n with j≠0 and n≠0. It follows that r+s = (i/j) + (m/n) = (a/b), which is a quotient of two integers, with a non-zero denominator. Hence it is a rational number. This was what was to be shown."

Topics in Section 4.3 (Direct Proof and Counterexample III: Divisibility)

* When n∈Z, and d∈Z with d≠0, d|n means "d divides n," which means that n=dm, where m∈Z. For example 2|10 because 10=2*5. We also express this idea by saying "n is a multiple of d," "d is a factor of n," and "d is a divisor of n."

* Divisors of zero: If d∈Z with d≠0, we say d is a "divisor of 0" because it is true that 0=d*0. (Every non-zero integer is divisor of 0.) For example 0=42*0, and 42≠0, so 42 is a divisor of 0.

* However 0 is not a divisor of anything.

* Theorem 4.3.1 - A Positive Divisor of a Positive Integer: For all integers m, n, if m and n are positive and m|n, then m≤n.
(Idea of the proof: n=mk, k∈Z. k must be ≥0. ∴ k≥1, so mk≥m, i.e. n≥m, qed)

* Theorem 4.3.2 - Divisors of 1: The only divisors of 1 are 1 and -1.
(Idea of the proof: 1*1=1 and (-1)*(-1)=1 shows that 1 and -1 are divisors of 1. If m is any divisor of 1, then 1=mk, m∈Z and k∈Z. If m and k are positive then by Theorem 4.3.1, 1≥m. The only positive integer ≤1 is 1, so

that means m is 1. If m and k are not both positive, they must both be negative. In that case, 1=(-m)(-k), where both -m and -k are positive. Again by Theorem 4.3.1, we can conclude that -m is 1, in other words m=-1.

* Theorem 4.3.3 - Transitivity of Divisibility: $\forall\ k \in Z,\ m \in Z,\ n \in Z$, if $k|m$ and $m|n$, then $k|n$

(Idea of proof: m=kd, n=mh, where $d \in Z$ and $h \in Z$. ∴ n=mh=(kd)h=k(dh). $dh \in Z$, so we have shown that $k|n$. qed)

* Theorem 4.3.4 - Divisibility by a Prime: Any integer n>1 is divisible by a prime number.

(Idea of proof: If n is prime, then it is divisible by a prime number - itself. If n is not prime then $n=r_0s_0$ where r_0 and s_0 are both integers "properly" between 1 and n. r_0 divides n. If r_0 is prime, then n is divisible by a prime. If r_0 is not prime then $r_0=r_1s_1$, where r_1 and s_1 are both integers "properly" between 1 and r_0. By the transitivity of divisibility, $r_1|n$. Also $1<r_1<r_0<n$. If r_1 is prime, then n is divisible by a prime. If not, r_1 can be factored as $r_1=r_2s_2$, where r_2 and s_2 are both integers "properly" between 1 and r_1, $r_2|n$ (by transitivity of divisibility), and $1<r_2<r_1<r_0<n$. If r_2 is prime, then n is divisible by a prime. This process of finding integers r_i has to stop with finding an r_i that is prime eventually, because if it did not, there would be an infinite <u>descending</u> sequence of integers greater than 1: $n>r_0>r_1>r_2>r_3> \ldots > 1$. This is a contradiction because, whatever positive integer n is, there are only finitely many integers between n and 1. Since the process stops with an r_i that is prime, and since that r_i divides n (by transitivity of divisibility), we have proved that n is divisible by a prime. qed)

* Theorem 4.3.5 - Unique Factorization of Integers Theorem (The Fundamental Theorem of Arithmetic): Given any integer n>1, n can be expressed as the product of a list of prime numbers. (In this kind of list, the same prime is allowed to appear multiple times.) Except for different possible orderings, there is only one such list for each n.

  Example: 24 = 2*2*2*3, and there is no other way to write 24 as a product of primes, except for shuffling the factors around like this: 24=2*2*3*2.

Sample illustrative problems for section 4.3

* 15: Prove directly from the definition of divisibility:
  For all integers a, b, and c, if $a|b$ and $a|c$, then $a|(b+c)$

* H19: Determine whether the statement is true or false,. If true, prove directly from the definitions. If false, give a counter example.

  For all integers a, b, and c, if a divides b then a divides bc.

* 20: Determine whether the statement is true or false,. If true, prove directly from the definitions. If false, give a counter example.

  The sum of any three consecutive integers is divisible by 3. (Integers m<n are consecutive if and only if n=m+1.)

* 31: Determine whether the statement is true or false,. If true, prove directly from the definitions. If false, give a counter example.

  For all integers a and b, if a|10b, then a|10 or a|b.

Topics in Section 4.4 (Proof and Counterexample IV: Division into Cases and the Quotient Remainder Theorem)

* The basic **idea of dividing** an integer j by another integer k is to 'represent j as some groups of size k'.

* For example, the idea of dividing 13 by 5 is to express 13 as two groups of five, with three left over.

```
        11111 11111 111  = 13 = 2*5+3
```

* Theorem 4.41 - The **Quotient Remainder Theorem:** Given any integer n and positive integer d, there exist unique integers q and r such that

  n = (d * q) + r and 0 <= r < d

Example:
```
 53 = 3 * 17 + 2;  So here q=17 and r=2 (0 <= r < d)
-53 = 3 * (-17) - 2 = 3 * (-18) + 1; So here q=(-18) and r=1 (0 <= r < d)
```
(When d is a factor of n, n=d*k for some integer k and the remainder r=0. In this case, -n=d*(-k) and the remainder is also zero. When n=d*k+r, with 0 < r < d (in other words r≠0) then the quotient remainder numbers for -n are:

```
           -n = d*(-k-1) + (d-r)
```

The example above of 53=3*17+2 and -53 =3*(-18)+1 illustrate the idea.

When n>0 and d>0, and n=dk+r (0<=r<d), k=n/d (k=n div d) and r=n%d (r = n mod d), where / and % are the C++ (or Java) operators.

* We can use Theorem 4.4.1 to **prove that every integer is either even or odd.**

(Idea of Proof: n=2q+r, where 0<=r<2. r must be 0 or 1. Therefore n is either even or odd. The uniqueness of q and r implies that n cannot be both even and odd.)

* Method of **Proof by Division into Cases**

To prove a statement of the form "If A_1, or A_2, or ... or A_n, then C," prove all of "If A_1 then C, if A_2 then C, ... and if A_n then C." The process shows that C is true regardless of which of A_1, A_2, ..., A_n happens to be the case.

An example of this is a proof that two consecutive integers have opposite parity. It's convenient to divide the proof into the two cases where the first integer is even, and the case where the first integer is odd.

* Theorem 4.4.6 - **The Triangle Inequality:** Let x, y be any two real numbers. |x+y|<=|x|+|y|.

* Lemma 4.4.4: For any real number r, -|r|<=r<=|r|.
  Proof: if r>=0 then the inequalities just say that -r<=r<r, which is obviously true.
  On the other hand if r<0, then the inequalities just say that r<=r<=-r, which is also obviously true.

(Idea of the proof of the triangle inequality:
From the lemma, we know that
x<=|x| and y<=|y| are true.
Therefore x+y <= |x|+|y|.

Case 1: x+y >= 0. In this case
|x+y| = x+y. Since x+y <= |x|+|y|,
we can conclude |x+y| <= |x|+|y|.

Case 2: x+y < 0. In this case
|x+y| = -(x+y) = (-x) + (-y).
We know from the lemma that
(-x) <= |-x| = |x|, and
(-y) <= |-y| = |y|.
Therefore (-x) + (-y) <= |x|+|y|.
Since, in this case
|x+y| = (-x) + (-y), we can also
conclude that |x+y| <= |x|+|y|.
)

Sample illustrative problems for section 4.4

* Problem 4.4.23: Prove that for all integers n, if n mod 5 = 3, then n^2 mod 5 = 4.
  (In other words if the remainder upon division of n by 5 is 3, then the remainder upon division of n^2 by 5 is 4.)

* Problem 4.4.51: If m, n, a, b, and d are integers, d>0 and m mod d = a and n mod d = b, is (m+n) mod d = (a+b)? Is (m+n) mod d = (a+b) mod d? Prove your answers.

Topics in Section 4.6 (Indirect Argument: Contradiction and Contraposition)

* The Method of 'Proof By Contradiction' (this method uses the 'rule of the excluded middle' - the idea that if a statement is NOT FALSE, then it must be TRUE, because there's presumably nothing 'in between' TRUE and FALSE.)
  1. If you want to prove S by contradiction, begin by making ~S, the logical negation of S, a PREMISE.
  2. Make logical deductions that lead to a contradiction - something that is known to be FALSE.
  3. Observe that ~S must be FALSE, because to assume it is true leads to a FALSE conclusion. Observe that S must therefore be TRUE. (~S is FALSE means "S is TRUE")

Example: Let $\{p_1, p_2, ..., p_n\}$ be a finite list of $n>=1$ prime numbers, where n is an integer.
Let $q = p_1*p_2*...*p_n + 1$,

in other words q is one more than the product f the list of primes.
Prove this statement S:
"None of the primes in the list is a divisor of q."

Proof: Assume ~S is true. In other words assume that there is a prime p in the list such that $p|q$, i.e. $q=pk$, where k is an integer.

Note also that p divides $p_1*p_2*...*p_n$
- in other words $p_1*p_2*...*p_n = ph$ where h is the product of all the primes in the list that are not equal to p. (If p is the only prime in the list, then h=1.) Since

$q = p_1*p_2*...*p_n + 1$,

$1 = q-p_1*p_2*...*p_n = pk-ph = p(k-h)$.

This shows that 1 is a multiple of p.

p is positive, by definition of prime, the product $p(k-h)= 1$ is positive. Therefore (k-h) is positive (it can't be negative or zero, because 1 is not negative or zero.)

$p>1$ by definition of a prime.

Since $p>1$ and (k-h) is positive, we can multiply $p>1$ and get $(k-h)p > (k-h)$.

Since (k-h) is positive, and (k-h) is an integer, $(k-h) \geq 1$, and so putting together some things we've proved:

$1=p(k-h)>(k-h)\geq 1$, by transitivity, $1>1$, which is a contradiction.

Since assuming ~S leads to the contradiction above, ~S must be FALSE.
In other words, S must be TRUE: $q = p_1*p_2*...*p_n + 1$, is not divisible by any of the primes $\{p_1, p_2, ..., p_n\}$. qed

Sample illustrative problems for section 4.6

* Carefully formulate the negations of the statement. Then prove the statement by contradiction.

4.6.11: ***S = "The product of any nonzero rational number and any irrational number is irrational."***

The negation ~S: "There exists a rational number $q\neq0$, and an irrational number s, such that the product qs is rational."

Proof of S:

Assume that ~S is true. $q = m/n$ for integers m and n with $m\neq0$ and $n\neq0$. (We know $m\neq0$ because $q\neq0$ is given.) Since qs is rational, we have
$(m/n)s = (i/j)$
where i and j are integers and $j\neq0$.
It follows that
$(n/m)(m/n)s = (n/m)(i/j)$,
which simplifies to
$s = (ni/mj)$.
We note that ni and mj are integers, and $mj\neq0$ can be concluded because $m\neq0$ and $n\neq0$.
Thus s is rational, by definition of rational. This contradicts the assumption of ~S, which included the assumption that s was irrational.
Therefore ~S must be FALSE, so S must be TRUE. qed.

* Prove by contraposition:

4.6.19: S = "If a product of two positive real numbers is greater than 100, then at least one of the numbers is greater than 10."

Proof: We attempt to prove the contrapositive of S: If two positive real numbers are both ≤ 10, then their product is ≤ 100. (This is logically equivalent to S.)
So suppose $0<a\leq10$ and $0<b\leq10$. Since $b>0$, multiplying through $a\leq10$ yields

$$ab\leq10b$$

Also multiplying through $b\leq10$ by 10 yields: $10b\leq100$.
Using transitivity, we get $ab\leq100$. qed.

Math 2300, Section #1; 14:00-14:50 MWR P-102; Daily Notes Fall, 2016

Topics in Section 5.1 (Sequences)

* Definition of a sequence
  A function F:D-->S from a set D to a set S, where D is either an 'integer interval' or an 'ascending integer ray'. In other words, the set D is either of the form D = $\{i \in Z \mid i \geq n\}$, where n is some integer, or of the form D = $\{i \in Z \mid m \geq i \geq n\}$, where m and n are integers, with m≥n.

  Example: $a_i = (-1)^i/(i!)$, $i \geq 1$ = -1, 1/2, -1/6, 1/24, -1/120, ...

* Definition of term, subscript, index, initial term, final term, subscript notation, definition of an ellipsis [...], infinite sequence, explicit formula, general formula (formulae for a general term of the sequence, expressed as a function of the subscript)

* An alternating sequence ... See the previous example.

* Finding an explicit formula to fit given initial terms

* Summation Notation (See notation in the inset for Theorem 5.1.1)

* expanded form, index of a summation, lower limit, upper limit
* Computing summations
* Summation terms given by a formula
* Changing between summation form and expanded form
* Separating off terms
* Telescoping sums
* Product notation (example below)

* Theorem 5.1.1: Properties of Summations and Products

Theorem 5.1.1

If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and c is any real number, then the following equations hold for any integer $n \geq m$:

1. $$\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n}(a_k + b_k)$$

2. $$c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$$ generalized distributive law

3. $$\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n}(a_k \cdot b_k).$$

* Change of variable - (example - the sum of k^3 from k=1 to k=3 is the same as the sum of j^3 from j=1 to j=3). [sums of consecutive cubes: 1, 9, 36, 100, ...]

(Another example of change of variable: the sum of 1/(k+1) from k=1 to k=n is the same as the sum of 1/h from h=2 to h=n+1.)

* Factorial and binomial coefficient ("n choose r") notation
* Sequences in Computer Programming
* Dummy variable in a loop
* Applications

Sample illustrative problems for section 5.1

* Write the first four terms of the sequences defined by the formula

5.1.3: $c_i = [(-1)^i]/3^i$ for all integers $i \geq 0$

* Write the first four terms of the sequences defined by the formula

5.1.5 $e_n = \lfloor n/2 \rfloor \cdot 2$ (2*floor(n/2) for all integers $n \geq 0$

* Find an explicit formula for the sequence

5.1.12: 1/4, 2/9, 3/16, 4/25, 5/36, 6/49

* Write using summation or product notation

5.1.46:

(2/(3*4))-(3/(4*5))+(4/(5*6))
-(5/(6*7))+(6/(7*8))

* Write using summation or product notation

5.1.50:
1/2! + 2/3! + 3/4! + ... + n/(n+1)!

Topics in Section 5.2 (Mathematical Induction I)

* The Principle of Mathematical Induction
  Let P(n) be a statement that is defined for integers n.
  Let c be a fixed integer. Suppose the following two statements are true:

  1. P(c) is true.
  2. For all integers $k \geq c$, if P(k) is true, then P(k+1) is true.

If so, then P(n) is true for all integers $n \geq c$

* A principle logically equivalent to the principle of mathematical induction:

Suppose S is any set of integers satisfying

1. $c \in S$
2. for all $k \geq c$ if $k \in S$ then $k+1 \in S$

Then S must contain every integer greater than or equal to c.

* (S is the set of all integers for which the theorem is true, according to the principle of mathematical induction)

* The basic step - proving P(c)

* The inductive hypothesis - the assumption that P(k) is true, $\exists\, k \in Z$, $k \geq c$

* The inductive step - proving that P(k) --> P(k+1)

* A ***closed form*** is a formula for the value of a sum that does not contain an ellipsis or a summation sign.

Example: $n(n+1)/2$ is a closed form for $1 + 2 + ... + n$.

Sample illustrative problems for section 5.2

* 5.2.10 Prove the statement by mathematical induction:

$1^2 + 2^2 + ... + n^2$
$= n(n+1)(2n+1)/6$

for all integers $n \geq 1$

* 5.2.13 Prove the statement by mathematical induction:

The sum from i=1 to n-1 of i(i+1) equals $n(n-1)(n+1)/3$ for integers $n \geq 3$

* 5.2.20 Use the formula for the sum of the first n integers and/or the formula for the sum of a geometric sequence to evaluate the sum, or to write it in closed form:

$4 + 8 + 12 + 16 + ... + 200$

* 5.2.22 Use the formula for the sum of the first n integers and/or the formula for the sum of a geometric sequence to evaluate the sum, or to write it in closed form:

$3 + 4 + 5 + 6 + ... + 1000$

* 5.2.27 Use the formula for the sum of the first n integers and/or the formula for the sum of a geometric sequence to evaluate the sum, or to write it in closed form:

$$5^3 + 5^4 + 5^5 + \ldots + 5^k, \text{ where } k \text{ is any integer with } k \geq 3.$$

Topics in Section 5.3 (Mathematical Induction II)

* Proving divisiblity of $2^{2n} - 1$ by 3

* Proving the inequality $2n+1 < 2^n$, for $n \geq 3$

* Proving a property of a sequence, e.g. when $a_1 = 2$, and $a_k = 5a_{k-1}$
  The problem is to show that $a_n = 2(5^{n-1})$ for $n \geq 1$.

* "A Problem with Trominoes"

Sample illustrative problems for section 5.3

* 5.3.6
* 5.3.10
* 5.3.21

Topics in Section 5.4 (Strong Mathematical Induction and the Well-Ordering Principle for the Integers)

* Principle of Strong Mathematical Induction

  Let P(n) be a property that is defined for integers n, and let a and b be fixed integers with $a \leq b$. Suppose the following two statements are true:

  1) P(a), P(a+1), ... and P(b) are all true (basis step)
  2) For any integer $k \geq b$, if P(i) is true for all integers i from a through k, then P(k+1) is true. (inductive step)

  Then the statement

  for all integers $n \geq a$, P(n)

  is true. (The supposition that P(i) is true for all integers i from a through k is called the inductive hypothesis. Another way to state the inductive hypothesis is to say that P(a), P(a+1), ... P(k) are all true.)

* Actually the Principle of Strong Mathematical Induction is not really any 'stronger' than the Principle of Mathematical Induction - anything that can be proved with one can also be proved with the other. However sometimes one or the other is more convenient for someone constructing a proof.

Math 2300, Section #1; 14:00-14:50 MWR P-102; Daily Notes Fall, 2016

* The Principle of Strong Mathematical Induction is also known as the second principle of induction, the second principle of finite induction, and the principle of complete induction.

* Using the Principle of Strong Mathematical Induction, we can conveniently prove that every integer greater than 1 is divisible by a prime.

* Another example is a proof of a property of a sequence

* Another example is proof that a multiplication of k factors always requires k-1 multiplications, regardless of how the factors are associated.

* Another example: Existence and uniqueness of binary integer representations

* The Well Ordering Principle is equivalent to both the ordinary and the strong principles of mathematical induction.

  The Well Ordering Principle:

  Every non-empty set of positive integers contains a least element.

* Equivalently, every non-empty set of integers that is bounded below contains a least element.

* Application: Proof of existence part of the Quotient Remainder Theorem.

(S = set of all non-negative integers of the form n-dk, where k is an integer)

Here is a proof of the existence part of the Quotient Remainder Theorem that uses only ordinary mathematical induction.

Theorem: $\forall$ n $\in$ Z, $\forall$ d, $1 \le d \in Z$, $\exists$ $q \in Z$, $\exists$ $r \in Z$, such that P(n) is true, where P(n) is this statement: $n=dq+r$, and $0 \le r < d$

Proof: Case #1, assume $n \ge 0$. We prove case #1 by induction.
(1) $0 = d*0+0$ establishes that P(0) is true with q=0, and r=0
(2) Suppose that P(k) is true, $\exists$ $0 \le k \in Z$.
Then $k = dh+s$, where $h \in Z$, $s \in Z$, and $0 \le s < d$.
It follows that $k+1=dh+(s+1)$. We know that $0<s+1 \le d$. If $s+1<d$, then $k+1=dh+(s+1)$ establishes that P(k+1) is true. If it is not true that $s+1<d$, then $s+1=d$. In that case, this is true: $k+1=dh+d=d(h+1)$, which shows that P(k+1) is true with q=h+1, and r=0.
We have demonstrated that P(k) --> P(k+1)
It follows from the principle of mathematical induction that P(n) is true for all integers $n \ge 0$.

Case #2: Assume $n<0$. By the proof above, P(-n) is true. Therefore

$-n = dp+t$, where $p \in Z$, $t \in Z$, and $0 \le t < d$

Therefore $n = d(-p)-t$, and $(-d)<(-t) \le 0$

If $t=0$, then this shows that $P(n)$ is true with $q=-p$, and $r=0$.
If $t \ne 0$, then $(-d)<(-t)<0$, and writing n this way

$n = d(-p-1)+(d-t)$,

we observe that $(-d)<(-t)<0$ implies $0<(d-t)<d$,

which shows $P(n)$ to be true with $q=(-p-1)$ and $r=(d-t)$.

This establishes that $P(n)$ is true for integers $n<0$. So cases #1 and #2 combined prove that $P(n)$ is true for all integers n, qed.

* Application: Proof that a strictly decreasing sequence of non-negative integers is finite.

Sample illustrative problems for section 5.4

* 5.4.7:
g(1)=3, g(2)=5,
g(k)=3g(k-1)-2g(k-2), ∀k, 3≤k∈Z

Prove: g(n)=2^n+1, ∀n, 1≤n∈Z

* 5.4.8: h(0)=1, h(1)=2, h(2)=3,
h(k)=h(k-1)+h(k-2)+h(k-3),
∀k, 3≤k∈Z

Prove: (a) h(n)≤3^n, ∀n, 0≤n∈Z

(b) Suppose s is any real number such that s^3 ≥ s^2+s+1 (Such a number s must be more than 1.83.)

Prove: h(n)≤s^n, ∀n, 2≤n∈Z

Topics in Section 5.6 (Defining Sequences Recursively)

* **A recurrence relation for a sequence a_0, a_1, a_2, ... is a formula that relates each term a_k to certain of its predecessors a_{k-1}, a_{k-2}, ..., a_{k-i}, where i is an integer with $k-i \ge 0$. The initial conditions for such a recurrence relation specify the values of a_0, a_1, a_2, ..., a_{i-1}, if i is a fixed integer, or a_0, a_1, ..., a_m, where m is an integer with $m \ge 0$, if i depends on k.**

* Computing Terms of a recursively defined sequence
E.g.

f(1)=1, f(2)=1, f(n)=f(n-1)+f(n-2)...
calculate f(4) and f(5).

* Writing a recurrence relation in more than one way
  E.g.
f(1)=1, f(2)=1, f(n)=f(n-1)+f(n-2)

h(0)=1, h(1)=1, h(k+1)=h(k)+h(k-1)

* (Different) Sequences that satisfy the same recurrence relation
E.g. The Fibonacci sequence satisfies
f(n) = f(n-1)+f(n-2) for n≥2
 The Lucas sequence satisfies the same relation. However the initial conditions are different.
 Fibonacci: f(0)=1, f(1)=1
 Lucas: f(0)=1, f(1)=3

* Showing that a sequence given by an Explicit Formula Satisfies a Certain Recurrence Relation
 E.g. Proof of the recursion relation C(k) = C(k-1)(4k-2)/(k+1) for the Catalan numbers

[C(n) = (2n choose n) * (1/(n+1))

* Examples of Recursively Defined Sequences
* The *recursive paradigm* (aka the *recursive leap of faith*)

* The Tower of Hanoi
* The Fibonacci Numbers
* Compound Interest
* Recursive Definitions of Sum and Product

Sample illustrative problems for section 5.6

* 5.6.3: Find the first four terms
  c(k) = k(c(k-1))^2, k≥1,; c(0)=1

* 5.6.8: Find the first four terms
v(k)=v(k-1)+v(k-2)+1, k≥3;
v(1)=1, v(2)=3

* 5.6.11: c(n)=2^n - 1, 0≤n, n in Z.
  Show c(k)=2c(k-1)+1 for 1≤k, k in Z

* 5.6.13: t(n)=2+n, n≥0; Show that t(k)=2t(k-1)-t(k-2), k≥3

Math 2300, Section #1; 14:00-14:50 MWR P-102; Daily Notes Fall, 2016

Topics in Section 5.7 (Solving Recurrence Relations by Iteration)

* Given a sequence that satisfies a ***recurrence relation***, a ***solution*** to the sequence is an ***explicit formula*** for each member of the sequence.

Example: The Fibonacci sequence
Recurrence relation: $F_1=1$, $F_2=1$, $F_n=F_{n-1}+F_{n-2}$, $\forall n$, $2\leq n\in Z$
Solution (explicit formula):

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$$

where:

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61803\,39887\cdots$$

* The ***Iteration Method*** of finding a solution: generate terms until you see a pattern, guess an explicit formula, and verify it.

Example: $a_0=1$, $a_n=a_{n-1}+2n+1$, $\forall n$, $1\leq n\in Z$
Solution by iteration: $a_1=a_0+2(1)+1=4$, $a_2=a_1+2(2)+1=9$,
$a_3=a_2+2(3)+1=16$, $a_4=a_3+2(4)+1=25$

So it looks like $a_n=(n+1)^2$ is an explicit formula for a_n.

The next step is usually to attempt to prove the guessed formula is correct, typically with proof by induction.

* Arithmetic sequence: $\{a_0, a_1, a_2, \ldots\}$, $\exists d\in Z$, $a_n=a_{n-1}+d$, $\forall n$, $1\leq n\in Z$
(solution: $a_n=a_0+dn$, $\forall n$, $0\leq n\in Z$)

* Geometric sequence: $\{a_0, a_1, a_2, \ldots\}$, $\exists d\in Z$, $a_n=da_{n-1}$, $\forall n$, $1\leq n\in Z$
(solution: $a_n=a_0d^n$, $\forall n$, $0\leq n\in Z$)
* other formulas:
  Arithmetic Series: $1+2+3+\ldots+n = n(n+1)/2$
  Geometric Series: $1+x+x^2+\ldots+x^n = (x^{n+1}-1)/(x-1)$; $0\neq x\in R$

Sample illustrative problems for section 5.7

* 5.7.1b: Find a formula for the expression $3+2+4+6+8+\ldots+2n$; $\forall n$, $1\leq n\in Z$

* 5.7.6: Use iteration to guess an explicit formula:
  $d_1=2$; $d_k=2d_{k-1}+3$, $\forall k$, $2\leq k\in Z$

* 5.7.10: Use iteration to guess an explicit formula:
  $h_0=1$; $h_k=2^k - h_{k-1}$, $\forall k$, $1\leq k\in Z$

* 5.7.35: Use mathematical induction to verify the correctness of the formula derived in exercise 5.7.10.

Topics in Section 5.8 (Second-Order Linear Homogeneous Recurrence Relations with Constant Coefficients)

* A second-order linear homogeneous recurrence relation with constant coefficients has the form

$$x_n=Ax_{n-1} + Bx_{n-2},$$

for all n greater than some fixed integer,

where A and $B\neq 0$ are fixed constant real numbers.

It's "second-order" because the value farthest back in the sequence used to define x_n is x_{n-2}.

It's "linear" because each term contains no more than one of x_n, x_{n-1}, or x_{n-2}: there are no second or higher powers of x_n, x_{n-1}, or x_{n-2}, and no products of two or more of x_n, x_{n-1}, or x_{n-2}.

It's "homogeneous" because every term has the same "degree" - the same total number of powers of any of x_n, x_{n-1}, or x_{n-2}. (When a recurrence relation is linear, the only way it can fail to be homogeneous is if it has a constant term.)

It has "constant coefficients" because A and B are fixed constants that do not depend on n.

* If t is some number such that

$\mathbf{t^2=At+B}$,

then

$t^k=At^{k-1}+Bt^{k-2}$

for all integers $k\geq 2$, which can be proved just by multiplying through the first equation by t multiple times.

* The equation $t^2=At+B$, and equivalently $\mathbf{t^2-At+B=0}$, is called the   of the relation. Lemma 5.8.1 says basically that the sequence $\{x_0=1,x_1=s,x_2=s^2,x_3=s^3,x_4=s^4,... \}$ satisfies the relation $x_n=Ax_{n-1} + Bx_{n-2}$ if and only if s is a root of the characteristic equation.

Example: Consider the recurrence relation $x_n=x_{n-1} + 2x_{n-2}$

The characteristic equation is $T^2-T-2=0$, which factors as $(T-2)(T+1)=0$. So the solutions are $T=2$ and $T=-1$. By lemma 5.8.1, the sequences

$\{1,2,2^2,2^3,...\}$ and $\{1,-1,(-1)^2,(-1)^3,...\}$ satisfy the recurrence, and are the ONLY sequences of powers that satisfy the recurrence.

* Any 'linear combination' of solutions to the recurrence $x_n=Ax_{n-1} + Bx_{n-2}$ is also a solution.

Example: We showed that these sequences $\{1,2,2^2,2^3,...\}$ and $\{1,-1,(-1)^2,(-1)^3,...\}$ are solutions to the recurrence, $x_n=x_{n-1} + 2x_{n-2}$

So, choosing any two arbitrary constants C and D, we can verify that

$\{C+D,\ 2C-D,\ 2^2D+(-1)^2C,\ 2^3D+(-1)^3C,\ ...\}$ is also a solution to the recurrence.

Proof: Suppose $x_n=Ax_{n-1} + Bx_{n-2}$ and $y_n=Ay_{n-1} + By_{n-2}$ for sequences

$\{x_n\}$ and $\{y_n\}$. Then Cx_n+Dy_n
$= C(Ax_{n-1} + Bx_{n-2}) + D(Ay_{n-1} + By_{n-2})$ // by the assumption
$= A\ (Cx_{n-1} + Dy_{n-1}) + B(Ax_{n-2} + Dy_{n-2})$ // by algebra

The latter term shows that the recursion relation holds for Cx_n+Dy_n

* If you have specific values z_0 and z_1 that you want for the first two terms of the solution sequence, and if you have two sequences $\{x_n\}$ and $\{y_n\}$ that satisfy the recurrence relation, you may be able to solve the two equations $Cx_0+Dy_0 = z_0$ and $Cx_1+Dy_1 = z_1$ for the values of C and D that give z_0 and z_1 as the first two terms of a solution.

* If s and t are two distinct solutions to the characteristic equation, then the sequences $\{1,s,s^2,s^3,s^4,...\}$ and $\{1,t,t^2,t^3,t^4,...\}$ are solutions to the recurrence, and the equations $C+D = z_0$ and $Cs+Dt = z_1$ can be solved to get any two desired numbers z_0 and z_1 for the first two terms of a solution sequence. Since $s\neq t$, the equations are guaranteed to have a solution.

* Since any solution to the recurrence is completely determined by the first two values and the recurrence relation, the information in the previous bullet indicates that all solutions to the recurrence are of the form $\{Cs^n+Dt^n\}$ when s and t are two distinct roots of the characteristic equation.

* Double root case: If the recurrence relation is $x_n=Ax_{n-1} + Bx_{n-2}$ and r is a double root of the characteristic equation $T^2-AT-B=0$, then $(T-r)^2$ is the characteristic equation, which is $T^2-2rT+r^2=0$. Thus $A=2r$, $B=-r^2$ and

$$\mathbf{A(k-1)r^{k-1} + B(k-2)r^{k-2}} = Akr^{k-1} + Bkr^{k-2} - Ar^{k-1} - 2Br^{k-2}$$
$$= kr^{k-2}(Ar+B) - r^{k-2}(Ar+2B)$$

Because r is a root of $T^2-AT-B=0$, $Ar+B=r^2$ and because $A=2r$ and $B=-r^2$, $Ar+2B=0$. Thus the last quantity in the chain of inequalities above is

$$kr^{k-2}(Ar+B) - r^{k-2}(Ar+2B) = \mathbf{kr^k}$$

This shows that the sequence ***$\{nr^n\}$ is a solution to the recurrence.*** (***$\{r^n\}$ is also a solution***, which can be shown in the same way as before.)

* Using the foregoing facts, it can be shown, with a routine proof, that when the characteristic equation has a double root, ***all solutions*** to the recurrence ***are linear combinations of $\{nr^n\}$ and $\{r^n\}$***, and a solution exists for every pair of initial values z_0 and z_1.

Sample illustrative problems for section 5.8

* Ex 5.8.1: Which of the examples are second-order linear homogeneous recurrence relations with constant coefficients?

* Ex 5.8.8: (a) find the sequences, based on roots of the characteristic equation, that are solutions to the recurrence relation; and (b) find an explicit formula that satisfies both the initial conditions and the recurrence relation.

* Ex 5.8.11: Find an explicit formula for the given recurrence relation and initial conditions.

* Ex 5.8.13: Find an explicit formula for the given recurrence relation and initial conditions.

Topics in Section 5.9 (General Recursive Definitions and Structural Induction)

* Recursive Definition of a Set:

  I. BASE: A statement that certain objects belong to the set
  Example: "Each symbol of the alphabet is a Boolean expression"

  II. RECURSION: A collection of rules indicating how to form new set objects from those already known to be in the set.
  Example: "If P and Q are Boolean expressions, then so are (a) $(P \wedge Q)$ and (b) $(P \vee Q)$ and (c) $\sim P$."

  III. RESTRICTION: A statement that no objects belong to the set other than those coming from I and II.
  Example: "There are no Boolean expressions over the alphabet other than those obtained from I and II."

* Definition of a ***string:*** Let S be a non-empty finite set. ***A string over S*** is a finite sequence of elements of S. The elements are called ***characters***

of the string, and the number of characters in the string is the ***length of the string.*** The ***null string*** (aka ***empty string***) is the string with no characters. The null string has **length 0** and is often denoted as **ε** (epsilon).

* ***The Structural Induction form of mathematical induction:***
  Let S be a set that has been defined recursively, and consider a proposition (statement) P(x) that may be true or false about objects x in S. To prove that P(x) is true for every x in S:
  1. Show that P(x) is true for every object x in the BASE of S.
  2. Show that for each rule in the RECURSION, if the rule is applied to an object x in S for which P(x) is true, then P(y) is true for the object y defined by the rule.

  Because no objects other than those obtained through the BASE and RECURSION conditions are contained in S, it must be the case that P(x) is true for every object x in S.

Sample illustrative problem for section 5.9

* 5.9.7: Define S recursively by

I. BASE: $\varepsilon \in S$

II. RECURSION: if $s \in S$, then
  $bs \in S$, $sb \in S$, $saa \in S$, and $aas \in S$

III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

Use structural induction to prove that every string in S contains an even number of a's.

Topics in Section 6.1 (Set Theory: Definitions and the Element Method of Proof)

* **The Element Argument: to prove that X ⊆ Y, where X and Y are sets:**
  1. Suppose x is a particular but arbitrarily chosen element of X, and
  2. Show that x is an element of Y

* **Set Equality:**
  Given sets A and B, A=B if and only if (A ⊆ B and B ⊆ A)

* **Set Operations**
  Suppose A and B are subsets of some 'universal set' U
  The union $A \cup B$ of A and B is the set of all elements of U that are in either A or B, or both.
  The intersection $A \cap B$ of A and B is the set of all elements that are in both A and B.
  The difference A-B is the set of all elements of A that are not elements of B.
  The complement A^c of A in U is the difference U-A.

Math 2300, Section #1; 14:00-14:50 MWR P-102; Daily Notes Fall, 2016

* **Interval Notation for real numbers:**
  Given a,b in R with $a \le b$
  $(a,b) = \{x \in R \mid a<x<b\}$
  $[a,b] = \{x \in R \mid a \le x \le b\}$
  $[a,b) = \{x \in R \mid a \le x<b\}$
  $(a,b] = \{x \in R \mid a<x \le b\}$
  $(a, \infty) = \{x \in R \mid a<x\}$
  $[a, \infty) = \{x \in R \mid a \le x\}$
  $(-\infty,b) = \{x \in R \mid x<b\}$
  $(-\infty,b] = \{x \in R \mid x \le b\}$

* **Unions and Intersections of an Indexed Collection of Sets**
* **Alternative Notations: e.g.** $A_0 \cup A_1 \cup \ldots \cup A_n$

• Definition

Unions and Intersections of an Indexed Collection of Sets

Given sets $A_0, A_1, A_2, \ldots$ that are subsets of a universal set U and given a nonnegative integer n,

$$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, \ldots, n\}$$

$$\bigcup_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for at least one nonnegative integer } i\}$$

$$\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for all } i = 0, 1, 2, \ldots, n\}$$

$$\bigcap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for all nonnegative integers } i\}.$$

* **The Empty Set.** The empty set is the set with no elements. There is only one empty set. It is denoted $\varnothing$.

* **Disjoint Sets.** Two sets are disjoint if their intersection is the empty set. Example: (1,2) and (3,4) are disjoint.

* **Pairwise Disjoint Collections of Sets** (aka **Mutually Disjoint**, aka **Non-overlapping**)**:** A collection of sets A_0, A_1, A_2, ... A_n is mutually disjoint if $A_i \cap A_j = \varnothing$ whenever $i \ne j$.

* **A Partition of Sets:** A finite or infinite collection of non-empty sets A_0, A_1, A_2, ... is a partition of a set A if and only if

1. A is the union of all the A_i.
2. The sets A_0, A_1, A_2, ... are pairwise disjoint.

* **Power Sets:** The power set $\mathcal{P}$(A) of the set A is the set of all subsets of A.

* **Cartesian Products**

• Definition

Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$ denoted $\boldsymbol{A_1 \times A_2 \times \ldots \times A_n}$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

In particular,

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$$

is the Cartesian product of A_1 and A_2.

• Definition

Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements. The **ordered $\boldsymbol{n}$-tuple, $\boldsymbol{(x_1, x_2, \ldots, x_n)}$,** consists of $x_1, x_2, \ldots, x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. An ordered 2-tuple is called an **ordered pair,** and an ordered 3-tuple is called an **ordered triple.**

Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are **equal** if, and only if, $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$.

Symbolically:

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \quad \Leftrightarrow \quad x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n.$$

In particular,

$$(a, b) = (c, d) \quad \Leftrightarrow \quad a = c \text{ and } b = d.$$

Math 2300, Section #1; 14:00-14:50 MWR P-102; Daily Notes Fall, 2016

Sample illustrative problem for section 6.1 (Definitions and the Element Method of Proof)

* Ex 6.1.4: A = integral multiples of 5, B = integral multiples of 20.
Is $A \subseteq B$? Is $B \subseteq A$? Explain

* Ex 6.1.11:
The universal set is the real numbers R.
The intervals A=(0,2]; B=[1,4); C=[3,9) are given.
Find various unions, intersections, and complements involving A,B,C.

(a) $(A \cup B)$ = ? ; (b) $(A \cap B)$ = ?; (c) (A^c) = ? (d) $(A \cup C)$ = ?;
(e) $(A \cap C)$ = ?; (f) (B^c) = ?; (g) $(A^c \cap B^c)$ = ?; (h) $(A^c \cup B^c)$ = ?;
(i) $((A \cap B)^c)$ = ?; (j) $((A \cup B)^c)$ = ?;

* Ex 6.1.15
Draw Venn Diagrams to describe sets satisfying certain conditions

(a) $(A \cap B) = \emptyset$, $A \subseteq C$, $(C \cap B) \neq \emptyset$
(b) $A \subseteq B$, $C \subseteq B$, $(A \cap C) \neq \emptyset$
(c) $(A \cap B) \neq \emptyset$, $(B \cap C) \neq \emptyset$, $(A \cap C) = \emptyset$, $A \nsubseteq B$, $C \nsubseteq B$

* 6.1.27 (a) & (d)
(a) Is {{a, d, e}, {b, c}, {d, f}} a partition of {a, b, c, d, e, f}?
(d) Is {{3,7,8}, {2,9}, {1,4,5}} a partition of {1,2,3,4,5,6,7,8,9}?

* 6.1.31: Find various power sets. A = {1,2} ; B = {2,3}
Find: $\mathcal{P}(A \cap B)$, $\mathcal{P}(A)$, $\mathcal{P}(A \cup B)$, $\mathcal{P}(A \times B)$,

Topics in Section 6.2 (Properties of Sets)

Theorem 6.2.1 Some Subset Relations

1. *Inclusion of Intersection:* For all sets A and B,

$$(a)\ A \cap B \subseteq A \quad \text{and} \quad (b)\ A \cap B \subseteq B.$$

2. *Inclusion in Union:* For all sets A and B,

$$(a)\ A \subseteq A \cup B \quad \text{and} \quad (b)\ B \subseteq A \cup B.$$

3. *Transitive Property of Subsets:* For all sets A, B, and C,

$$\text{if } A \subseteq B \text{ and } B \subseteq C, \text{ then } A \subseteq C.$$

Procedural Versions of Set Definitions

Let X and Y be subsets of a universal set U and suppose x and y are elements of U.

1. $x \in X \cup Y \quad \Leftrightarrow \quad x \in X \text{ or } x \in Y$
2. $x \in X \cap Y \quad \Leftrightarrow \quad x \in X \text{ and } x \in Y$
3. $x \in X - Y \quad \Leftrightarrow \quad x \in X \text{ and } x \notin Y$
4. $x \in X^c \quad \Leftrightarrow \quad x \notin X$
5. $(x, y) \in X \times Y \quad \Leftrightarrow \quad x \in X \text{ and } y \in Y$

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,

$$\text{(a) } A \cup B = B \cup A \quad \text{and} \quad \text{(b) } A \cap B = B \cap A.$$

2. *Associative Laws:* For all sets A, B, and C,

$$\text{(a) } (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$
$$\text{(b) } (A \cap B) \cap C = A \cap (B \cap C).$$

3. *Distributive Laws:* For all sets, A, B, and C,

$$\text{(a) } A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$
$$\text{(b) } A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

4. *Identity Laws:* For all sets A,

$$\text{(a) } A \cup \emptyset = A \quad \text{and} \quad \text{(b) } A \cap U = A.$$

5. *Complement Laws:*

$$\text{(a) } A \cup A^c = U \quad \text{and} \quad \text{(b) } A \cap A^c = \emptyset.$$

6. *Double Complement Law:* For all sets A,

$$(A^c)^c = A.$$

7. *Idempotent Laws:* For all sets A,

$$\text{(a) } A \cup A = A \quad \text{and} \quad \text{(b) } A \cap A = A.$$

8. *Universal Bound Laws:* For all sets A,

$$\text{(a) } A \cup U = U \quad \text{and} \quad \text{(b) } A \cap \emptyset = \emptyset.$$

9. *De Morgan's Laws:* For all sets A and B,

$$\text{(a) } (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b) } (A \cap B)^c = A^c \cup B^c.$$

10. *Absorption Laws:* For all sets A and B,

$$\text{(a) } A \cup (A \cap B) = A \quad \text{and} \quad \text{(b) } A \cap (A \cup B) = A.$$

11. *Complements of U and $\emptyset$:*

$$\text{(a) } U^c = \emptyset \quad \text{and} \quad \text{(b) } \emptyset^c = U.$$

12. *Set Difference Law:* For all sets A and B,

$$A - B = A \cap B^c.$$

* How to prove two sets A, B are equal: Prove ($A \subseteq B$) and ($B \subseteq A$)
* Theorem 6.2.4: A set with no elements is a subset of very set.
* There is only one set with no elements, <u>the</u> empty set.

* How to prove a set is empty by contradiction: Assume it has an element and derive a contradiction.

Sample illustrative problem for section 6.2 (Properties of Sets)

* 6.2.3: Fill in the blanks of the following proof that $\forall$ sets A, B, C, when ($A \subseteq B$) and ($B \subseteq C$), ($A \subseteq C$).

Proof: Suppose A, B, and C are sets, and that ($A \subseteq B$) and ($B \subseteq C$). To show that ($A \subseteq C$), we must show that every element in ____(a)____ is in ___(b)____. But given any element in A, that element is in ____(c)_____ (because $A \subseteq B$), and so that element is also in ____(d)____ (because ____(e)____). Hence ($A \subseteq C$).

* 6.2.10: Use an element argument to prove the statement. Assume all sets are subsets of a universal set U.

For all sets A, B, and C, $(A - B) \cap (C - B) = (A \cap C) - B$

* 6.2.16: Use an element argument to prove the statement. Assume all sets are subsets of a universal set U.

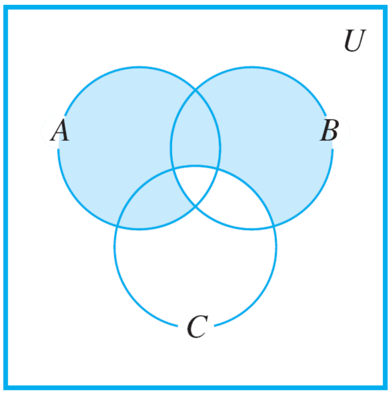
For all sets A, B, and C, if ($A \subseteq B$) and ($A \subseteq C$) then ($A \subseteq (B \cap C)$)

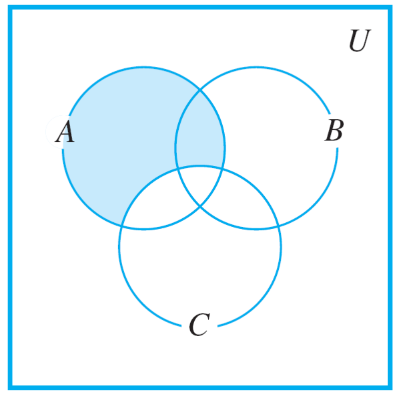
* 6.2.34: Use the element method for proving the set equals the empty set. (i.e. Suppose the set has an element and derive a contradiction.) Assume that all sets are subsets of a universal set U.

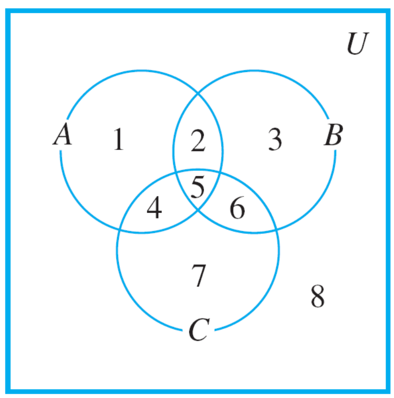
For all sets A, B, and C, if $(B \cap C) \subseteq A$, then $(C - A) \cap (B - A) = \emptyset$

Topics in Section 6.3 (Disproofs, Algebraic Proofs, and Boolean Algebras)

* A way to construct a counter example to a false set identity







We check the diagram to ascertain the plausibility that

$(A - B) \cup (B - C) = A - C$

is a correct set identity. Using the labels of the sections to construct example sets, we get $A = \{1,2,4,5\}$, $B = \{2,3,5,6\}$, and $C=\{4,5,6,7\}$. Here $A - C = \{1,2\}$, $A - B = \{1,4\}$, and $B - C = \{2,3\}$

* Proof by induction on $0 \le n \in \mathbb{Z}$ that a set with n elements has 2^n subsets.

Sample illustrative problem for section 6.3 (Disproofs, Algebraic Proofs, and Boolean Algebras)

* 6.3.1: Find a counter-example: $\forall$ sets A, B, and C, (all contained in a universal set, U) $(A \cap B) \cup C = A \cap (B \cup C)$

* 6.3.17: Prove or give a counter-example: ∀ sets A, B (contained in a universal set U): if A ⊂ B then $\mathcal{P}(A) \subset \mathcal{P}(B)$.

* 6.3.18: Prove or give a counter-example: ∀ sets A, B (contained in a universal set U):

* 6.3.34: Construct an algebraic proof (a proof that uses only the identities of theorem 6.2.2.)
  Prove: ∀ sets A, B, C (contained in a universal set U):
  (A-B)-C = A-(B ∪ C)

Topics in Section 8.1 (Relations on Sets)

* Examples of binary relations: x≤y in RxR, and "m-n is even" in ZxZ.
* The inverse of a binary relation: If R is a relation on AxB, then R^{-1} is the relation on BxA given by (b,a) ∈ R^{-1} if and only if (a,b)∈ R.
* A relation on a set A is a subset of AxA. It can be represented as a directed graph with a set of nodes representing the elements of A.
* An N-ary relation - a relation on the cross product of N sets - is a subset of that cross product. As objects of logical and mathematical study, relational databases are viewed as N-ary relations.

Sample illustrative problem for section 8.1 (Relations on Sets)

* 8.1.1: The congruence relation E from Z to Z is defined by m E n <--> (m-n) is even, ∀ m,n ∈ Z.
  a) Which of these are true? 0E0, 5E2, (6,6)∈E, (-1,7)∈E
  b) Prove that nE0 is true for any even integer n.

* 8.1.9: Let A be the set of all strings of length 4 made of 0's 1's and 2's. Define the relation R on A by: sRt <--> the sum of the characters in s is equal to the sum of the characters in t.
  a) is 0121 R 2200? What about 1011 R 2101, 2212 R 2121, and 1220 R 2111?

* 8.1.10: Let A = {3,4,5} and B = {4,5,6} and let R be the "less than" relation on AxB. State explicitly which ordered pairs are in R and R^{-1}.

Topics in Section 8.2 (Reflexivity, Symmetry, and Transitivity)

* Definition of reflexive relation on a set, symmetric relation on a set, and transitive relation on a set
* The relation of "equality" on the set of real numbers is reflexive, symmetric, and transitive. (Any 'reasonably defined' equality relation will be reflexive, symmetric, and transitive. For example equality of sets is reflexive, symmetric, and transitive.)

* The "less than" relation on real numbers is transitive. The "less than or equal" relation on real numbers and the set containment relation are reflexive and transitive, but not symmetric.

* Congruence of integers modulo n is reflexive, symmetric, and transitive.

* Given a relation that is reflexive and symmetric, the relation can be extended to its "transitive closure" -- by adding all the necessary pairs to the relation to make it transitive.

Sample illustrative problem for section 8.2 (Reflexivity, Symmetry, and Transitivity)

8.2.2

8.2.4

8.2.21

8.2.22

8.3.21

Topics in Section 8.3 (Equivalence Relations)

* **The Relation Induced by a Partition:** Given a partition of a set U (a collection of pairwise disjoint subsets of U whose union is U), we can say x is related to y if x and y are both elements of the same subset of the partition.

* **Theorem 8.3.1:** A relation R induced by a partition of a set U is reflexive, symmetric, and transitive.

* **Definition:** Suppose R is a relation on a set U. R is called *an equivalence relation if it is reflexive, symmetric, and transitive.*

* **Definition:** If R is an equivalence relation on a set U, and if $a \in U$, then *the equivalence class* $[a] = \{b \in U \mid aRb\}$.

* **Theorem 8.3.4:** The Partition Induced by an Equivalence Relation: If U is a set and R is an equivalence relation on U, then the distinct equivalence classes of R form a partition of U; that is, the union of the equivalence classes is all of U, and the intersection of any two distinct classes is empty.

* **Definition:** if S is one of the equivalence classes of a relation R on a set U, and if $x \in S$, we say that *x is a representative of S*.

Sample illustrative problem for section 8.3 (Equivalence Relations)

8.3.6

8.3.21

Topics in Section 9.4 (The Pigeonhole Principle)

* **The Pigeonhole Principle:** A function from one finite set into a smaller finite set cannot be one-to-one: There must be a least two elements in the domain that have the same image in the co-domain.

* **The Generalized Pigeonhole Principle:** For any function f from a finite set X with n elements to a finite set Y with m elements, and for any positive integer k, if $k < (n/m)$, then there is some $y \in Y$ such that y is the image of at least k+1 distinct elements of X. (The number of pigeons in every pigeonhole can't be less than the average, n/m, number of pigeons per pigeonhole.)

Sample illustrative problem for section 9.4 (The Pigeonhole Principle)

9.4.4

9.4.27

Possible Final Exam Topics

* Relations
* Logical Statements and Statement Forms
* Negations of various kinds of statements
* Conditional statements, converses, contrapositives, and inverses
* Valid and invalid arguments
* Existential and Universal statements
* Second-order linear homogeneous recurrence relations with constant coefficients
* Elementary proofs using basic definitions and properties
* Proof by contradiction
* Proof by induction
* Properties of sets and power sets
* Relations on a set.
* Reflexivity, Symmetry, and Transitivity
* Equivalence Relations and equivalence classes
* Partitions of a set
* The Pigeonhole Principle