**Math 2300, Section #1; 14:00-15:15 MW P-101; Daily Notes  Fall, 2015**

* **Take Roll**
   *  Tell students that they are required to carefully read the entire
   course description on line
    (or request a printed full copy if not able to read on  line)

* **Go over items in course description**
   + Name
   + Office & Hours
   + E-mail
   + Homepage + Class web site
* Go over Course Objectives - what do folks think those objectives mean?
* **Text**
   + Problems getting one?
   + How many will be using Ed #4? How many using Ed #3?
   + Do students know about the book web site - go and check out the errata
   and review materials with the class

* **Course components**
   + HW (25%)
   + 2 or more quizzes (50%)
   + comprehensive final exam (25%)
   *Exception:  You fail the course if you get a failing average on tests.*
   *Otherwise I use the weights above.*

**HOMEWORK**
**Look at the Assignments with Class -- First One is TODAY**

   Make sure to be familiar with homework submission rules in the course
   description -- **notice strict late policy** -- anybody here not able to
   access the course description, ask me for a paper copy.
**Schedule - look at the schedule with the class and go over this week's**
   **assignments -- There is reading and homework.  The first HW is due next**
   **Monday.**


**Lecture**
**Chapter One - Chapter One is about language we use to make it easier to talk**
   **about mathematical concepts, and to make it easier to find solutions to**
   **math problems.**

Section 1.1 - Variables
Section 1.2 - The Language of Sets
Section 1.3 - The Language of Relations and Functions

Section 1.1 - Variables are names we use for numbers or other things.

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         Example: (x)
      We can use a variable as a names of a specific numbers
      that we don't know.
       Example: x = 23981 times 53204
      When we want to state a fact that is true about a lot of
      numbers, we can use a variable as a name that represents
      any one of those numbers.
       Example: 2(y + 1) - y = y + 2
```

If we are wondering if a number with a certain property exists, we can use a
   variable to represent the number, to help us find the number, or to prove
   it does NOT exist.

Example: Can 6 be the sum of a number and its square?
With variable:  x^2 + x = 6.  Using quadratic formula, we find that 2 and -3
   work.

It's handy to use variables as names for numbers if we write some
   mathematical information that is long.  We can use a variable name to
   refer to something over and over again in different parts of what we
   write.  The reader understands what we mean, because we use the variable
   name consistently.

People often use variables to make mathematical statements
Universal: All real numbers y have the property that 2(y + 1) - y = y + 2
Conditional: If x=2 or x=-3 then x^2 + x = 6
Existential: There are two numbers x,y such that x^2 = 25, and y^2 = 25.

**Work problems with the class: 1.3, 1.10**

Section 1.2 - The Language of sets

Sets are basically just collections of things - kind of like a basket,
   containing some things, or maybe empty.

The things that are in the set are called elements of the set.
If S is a set and x is an element of S, we use the notation x ∈ S to denote
   that.

We can indicate the elements of a set by using "roster" notation with or
   without ellipsis:

S={a, t, r} ; T= {1, 2, ..., 14}; V = {31.2, 41.2, 51.2, ... }

When you use ellipsis you need to be sure that the people who read your
   writings will understand what you mean by the ellipsis.  That's why
   something is written in the beginning to establish the pattern.

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The idea of a set does not include any notion of the order of the elements.
S={1,2,3}; T={2,1,3}; and U={2,3,1} are all the same set, because the
   elements are the same.

Also, an element is either in the set or not.  An element can't be in a set
   twice.
For example {1,2,2} = {1,2}

Familiar sets R, Z, Q, R+, Z-, etc

R and Z illustrate the difference between the idea of continuous and discrete
   mathematics

Tell students to work as many HW problems as possible between now and
   Wednesday, and come in with their questions about how to work them.

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Day Two: Wed, Aug 26, 2015
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*  Take Roll
Today's HW assignment is problems for section 1.2 on page 13: 2,4,7,9,12; Due
   on Wednesday, September 2.
* Mondays assignment is 1.1 problems on page 5, 2,4,7,11,13; due Monday, Aug
     31.
*  The goal today is to finish discussion of chapter one.

Ask students for questions about the assigned HW.

**Show** class some example of "**set builder**" **notation** – say using inequality
   conditions
There's roster notation to specify a set – like **S = {-1, 0, 1, 2, 3}.**
You can also use "SET BUILDER" notation to specify a set:

For example the set S above is also described as  **S = { n ∈ Z | –1 ≤ n ≤ 3 }.**
It's the same set – You read the notation this way:

"**The set of all n in Z (integers) such that n is between –1 and 3
   (inclusive).**

The **set builder notation is generally <u>more expressive</u>** than the roster
   notation.  In other words, it's often easier to say what you mean using
   the set builder notation.  For **example:**

**C1 = { m ∈ Z | m = pq, where p and q are primes}**

Imagine trying to use roster notation to specify the set C1.

Notions of **subset**, **proper subset**, **equal sets**, **difference between** being a
   **subset and** being an **element**.

**Subset: if A and B are sets**, then **A ⊆ B means** that **every element of A is also
   an element of B**

Example: {2, 3, 1} ⊆ {-20, 2.3, 1, 3, 2, 4, 18}
Example: Z ⊆ R
Example: {1,2} ⊆ {1,2}

**Proper Subset:** if A is a subset of B, and if **there is at least one element of
   B that is NOT in A**, then A is a proper subset of B.

Example of Proper Subset Relation: {2, 3, 1} ⊆ {-20, 2.3, 1, 3, 2, 4, 18}
Example of Proper Subset Relation: Z ⊆ R

**Equality of Sets:**  If A and B are sets such that **A ⊆ B AND B ⊆ A**, then A and
   B are equal sets.  In that case every element of A is an element of B, and
   every element of B is an element of A.

Example: {7, 7, 2, 3, 1} = {1, 1, 2, 3, 7}

**Difference between being an element and being a subset.**
⊆  and ∈ **mean completely different things.**
For example {2, 5} ⊆ {5, 2, 4} but {2, 5} is not an element of {5, 2, 4}.
   The only elements of {5, 2, 4} are integers, and {2, 5} is not an integer.
Another example: a singleton set is not the same as its element:  4 and {4}
   are not the same thing.  4 ∈ {4}, but neither is a subset of the other.
The empty set is the set with no elements.  ∅

**The empty set is a subset of every set.**  This fact is **"vacuously" true.**
If S is a set, then  ∅ ⊆ S is true.  **Every element of ∅ is an element of S.**
That's a vacuously true statement, because the empty set has no elements.

**A slightly different way of looking at this: If ∅ is NOT** a subset of every
   set, then **there must be some set K such that ∅ is NOT a subset of K.**  For
   THAT to be true, **there must be an element of ∅ that is not an element of
   K.**  But since there are NO elements of ∅, there isn't one that's not in K.

**Idea of Cartesian product,** how to express ordered pairs as sets, and as
   "tuples"

Named after mathematician Rene Descartes, the Cartesian product of a set A
   and a set B is

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**A x B = { (a,b) | a ∈ A and b ∈ B }**

(a,b) is called an <u>ordered pair.</u>  It is not the same as the set {a,b} because
   the it has **a first element a, and a second element b.**  So although {a,b} =
{b,a}, it is not the case in general that **(a,b) = (b,a).**  (a,b) = (b,a) only
   when a = b.

When mathematicians need **to be very precise** about what they mean by an
   ordered pair, they can use set notation, and this definition:

**(a,b) is defined as { {a}, {a,b} }**

When a ≠ b, **a is distinguished** as the 'first' element of the ordered pair **by
   the fact that it is in both of these sets: {a}  {a,b}.**

**(a,b) = (c,d) iff a=c and b=d.**

Work problems with the class: 2.3, 2.5 (show the relation between A and E),
   2.8, 2.10, 2.11
(2.11)Set-roster notation
(2.3b,c)Sets are not multi-sets
(2.5)R, Z, Q, etc
(2.5)Set-builder notation
(2.8)Subsets
(2.8)Proper Subsets
(2.5)Equality of sets
(2.3a,c) Distinction between element and subset relations
(2.11)Cartesian products
ordered pairs
(2.10a,b,c)equality of ordered pairs
Set notation for ordered pair

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Class work on Section 1.3

* (3.1,3.5) Defining a relation between elements of A and elements of B as a
   subset of the cartesian product A x B.
* (3.1c) Domain of a relation (A in A x B)
* (3.1c,3.13) Co-domain of a relation (B in A x B)
*(3.1d,3.13) Arrow diagram of a relationGraph of a relation
* (3.5,3.11) Graph of a relation
* (3.11,3.13) Functions
* (3.11,3.13) Functions as types of relations
   (exactly one arrow from each domain element)
* (3.13) x, F(x) notation for function F
* (3.11) vertical line test to determine if a 'graph' is a function
* Functions as machines - black boxes

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* Equality of functions - pointwise

Sample problems to work:
3.1, 3.5, 3.11, 3.13


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Day Three: Mon, Aug 31, 2015
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*  Take Roll

Finish up with problems 3.5, 3.11, and 3.13 from last week.

Topics in Section 2.1
+ (3)(Forms of) Logical arguments
+ (J1) Statements (i.e. propositions)
+ (6) Using ~, ∨, ∧ to make compound statements
+ (32) Expressing inequalities using and, or
+ (16,18) truth values
+ (6) conjunction p ∧ q
+ (6) disjunction p ∨ q
+ Evaluating compound statements in general
+ Exclusive OR
+ (16,18) Logical Equivalence - same truth values for all substitutions of
    values for variables.
+ (30,32) Negations of conjunctions and disjunctions - De Morgan's Laws
+ (40,41) Tautologies - always true no matter the value of variables
+ (40,41) Contradictions - always false no matter the value of variables
+ (48) Laws of Boolean algebra (e.g. commutative, associative, ... )
+ (48) simplifying statements using Boolean algebra

Problems to work

2.1.3

J1: Which of the following is a statement?
    (a) That's what I like about you.
    (b) Tupelo is a kind of tree.

2.1.6, 2.1.16, 2.1.18, 2.1.30, 2.1.32, 2.1.40, 2.1.41, 2.1.48


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Day Four: Wed, Sep 02, 2015
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2.1.32 use one of De Morgan's laws to write the negation of
      $-2 < x < 7$

2.1.40 tautology or contradiction?
       $(p \land q) \lor (\sim p \lor (p \land \sim q))$

2.1.41 tautology or contradiction?
       $(p \land \sim q) \land (\sim p \lor q)$

2.1.48 A logical equivalence is derived below. Supply a reason for each step.
   (See p. 35 of Epp for the list of equivalences.)

$(p \land \sim q) \lor (p \land q) \equiv p \land (\sim q \lor q)$ by  _(a)_

                    $\equiv p \land (q \lor \sim q)$ by  _(b)_

                    $\equiv p \land \mathbf{t}$ by  _(c)_

                    $\equiv p$  by  _(d)_


Topics in Section 2.2 (Conditional Statements)
* Hypotheses (antecedents), conclusions (consequents), and conditional
     statements
* Precedence of --> among  $\lor$, $\land$ , and $\sim$  (--->last, $\sim$ first)
* Truth tables for conditional statements
* Representation of if-then as OR  $((p\text{-->}q ) \equiv (\sim p \lor q) )$
* Negation of conditional statement  $( \sim(\sim p \lor q) \equiv (p \land \sim q) )$
* Equivalence of a conditional statement and its contrapositive
       $( (p\text{-->}q ) \equiv (\sim q \text{ --> } \sim p) )$
   Maybe the easiest thing is to start with examples of contrapositives.
* Converse and inverse of a conditional statement [Conditional (p-->q)],
     [Inverse of the Conditional: ($\sim$p-->$\sim$q)], [Converse of the Conditional
     (q-->p)]
* Only if and the bi-conditional.  In logic the phrase p only if q is
     logically equivalent to p --> q
* If and only if p <--> q is logically equivalent to (p-->q) $\land$ (q-->p)
* Necessary and sufficient conditions
     "r is sufficient for s" means r-->s
     "r is necessary for s" means ($\sim$r --> $\sim$s) $\equiv$ (s-->r)
     "r is necessary and sufficient for s" means (s<-->r), which is the same
   meaning as s if and only if r.

Sample illustrative problems:

2.2.2 Re-write in if-then form: "I am on time for work if I catch the 8:05
   bus"

J2: Bear to the right or you'll get into a collision

2.2.16 Write the two statements in symbolic form and determine whether they
       are logically equivalent.  Include a truth table and a few words of
       explanation.
If you paid full price, you didn't buy it at Crown Books.
You didn't buy it at Crown Books or you paid full price.

2.2.19 Write the negation of "If Sue is Luiz' mother, then Ali is his
       cousin."

2.2.24 Use truth tables to establish the truth of this statement:
       "A conditional statement is NOT logically equivalent to its converse."

2.2.32 Rewrite as a conjunction of two if-then statements:
       "This quadratic equation has two distinct real roots if, and only if,
       its discriminant is greater than zero"

2.2.34 Rewrite the statement in if-then form in two ways, one of which is the
       contrapositive of the other.
       "The Cubs will win the pennant only if they win tomorrow's game."

2.2.40 Rewrite in if-then form: "Catching the 8:05 bus is a sufficient
       condition for my being on time for work."


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**Day Five: Wed, Sep 09, 2015**
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* **Take Roll**
* **Handback HWs #1 and #2**
* **Check the HW due -- sections 2.1 and 2.2**
* **The next homework due is on section 2.3 – Due MONDAY, Sept 14**


**Note: We won't cover digital circuits in this course, but you will see
    applications of formal logic when you study digital circuits (mainly in
    Computer Organization – CS 3740)**

Topics in Section 2.3 (Valid and Invalid Arguments)

* An argument is a series of statements – the last statement is called the
   conclusion, and the others are called premises
* Valid argument – conclusion must be true if premises are true
   in other words it is impossible for the conclusion to be false when the
   premises are true.
* Inferred, deduced
* Testing an argument for validity with a truth table
* Critical row of a truth table

* Syllogisms(two premises + conclusion), major premise, minor premise
* Modus Ponens and Modus Tollens
* Rules of inference
* Generalization (p ∴ p ∨ q; q ∴ p ∨ q)
* Specialization: (p ∧ q, ∴ p; p ∧ q, ∴ q)
* Elimination (p ∨ q, ~q, ∴ p; p ∨ q, ~p, ∴ q)
* Transitivity (p-->q, q-->r, ∴ p-->r)
* Proof by division into cases (p ∨ q, p-->r, q-->r, ∴ r)
* Fallacies
* The converse error (the fallacy of affirming the consequent)
* The inverse error (the fallacy of denying the antecedent)
* Sound argument - valid and all premises true
* Unsound argument - any argument that is not sound
* Contradiction Rule  (~p-->**c**, [c is a contradiction] ∴ p )

Sample illustrative problems:

2.3.8: Use truth tables to determine whether  (p ∨ q, p-->~q, p-->r, ∴ r) is
   valid.  Indicate which columns represent premises, and which represents
   the conclusion.  Explain how the truth table supports your answer.  Show
   you understand the definitions of valid/invalid.

2.3.22: Use symbols to write the logical form of the argument.  Then use a
   truth table to test the argument for validity.  Indicate which columns
   indicate the premises and which represents the conclusion. Include words
   of explanation showing you understand the meaning of validity.

If Tom is not on Team A, then Hua is on Team B.
If Hua is not on Team B, then Tom is on Team A.
∴ Tom is not on Team A, or Hua is not on Team B.

2.3.27: Write the logical form of the argument with symbols.  If valid,
   identify the rule of inference that assures its validity.  Otherwise state
   whether the inverse or converse error is made.

If this number is larger than 2, then its square is larger than 4.
This number is not larger than 2.
∴ The square of this number is not larger than 4.

Let p be "This number is larger than 2" and let q be "its square is larger
   than 4"

In symbols the argument says

p --> q
~p
∴ ~q

Topics in Section 3.1 (Predicates and Quantified Statements I)

* **Predicate calculus** (the symbolic analysis of predicates and quantified
  statements)
* **Statement calculus** (also: propositional calculus - the symbolic analysis of
  ordinary compound statements)
* **Predicate symbol** - obtained by removing some or all nouns from a statement
  (for example: "is a student at Bedford College", or "is a student at")
* **Predicate variables** - "when concrete values are substituted in place of
  predicate variables, a statement results." -- for example x and y are the
  predicate variables here:  "x is a student at y".
* **Predicate** (also: propositional function or open sentence - a predicate
  symbol together with suitable predicate variables)
* **Domain of a predicate variable** (the set of all values that may be
  substituted in place of the variable)
* **Truth set of a predicate P(x)** (the set of all values x in the domain of the
  variable that make P(x) true - $\{x \in D \mid P(x)\}$)
* **The Universal Quantifier** $\forall$ (for all, for each)
* **Truth or Falsity of a Universal Statement** (true if and only if substitution
  of every element of the domain of the variable(s) yields a true statement)
* **The Existential Quantifier** $\exists$ (there exists, for some)
* **Truth or Falsity of an Existential Statement** (true if and only if the
  substitution of at least one value in the domain make a true statement)
* **Universal Conditional Statements** (for example: $\forall x, P(x) \rightarrow Q(x)$)
* **Equivalent Forms of Universal and Existential Statements**
* **Implicit Quantification** (for example: "if $n \in Z$ then $n \in Q$", or "24 can be
    written as the sum of two even integers")

Sample illustrative problems:

3.1.5: Let Q(x,y) be the predicate "If x<y then $x^2 < y^2$" with domain for both x
   and y being the set **R** of real numbers.
a. Explain why Q(x,y) is false if x = -2 and y=1.
b. Give values different from those in part (a) for which Q(x,y) is false.
c. Explain why Q(x,y) is true if x=3, and y=8.
d. Give values different from those in part (c) for which Q(x,y) is true.

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**Day Six: Mon, Sep 14, 2015**
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* **Take Roll**
* **Handback HWs??**
* **Collect HW on Section 2.3 due today**
* **The next homework due is on section 3.1, due Wednesday, Sept 16**

3.1.13: Consider the following statement:

∀ basketball players x, x is tall.

Which of the following are equivalent ways of expressing this statement?

a. Every basketball player is tall.
b. Among all the basketball players, some are tall.
c. Some of all the tall people are basketball players.
d. Anyone who is tall is a basketball player.
e. All people who are basketball players are tall.
f. Anyone who is a basketball player is a tall person.

3.1.16a: Rewrite in the form "∀ _____ x, _____"
"All dinosaurs are extinct."

3.1.16c: Rewrite in the form "∀ _____ x, _____"
"No irrational numbers are integers."

3.1.16e: Rewrite in the form "∀ _____ x, _____"
"The number 2,147,581,953 is not equal to the square of any integer."

3.1.32: Let R be the domain of the predicate variable x. Which of the
  following are true and which are false? Give counter examples for the
  statements that are false.
(a) $(x>0) \Rightarrow (x>1)$

(b)
(c) $(x^2 > 4) \Rightarrow (x > 2)$


Topics in Section 3.2 (Predicates and Quantified Statements II)

* **Negating Quantified Statements:** ( ~(∀ x∈D, P(x)) ≡ (∃ x∈D, ~P(x)) )
* **Relation among** ∀, ∃, ∨, ∧
    When D = {$x_1$, $x_2$, ..., $x_n$} (∀ x∈D, P(x)) ≡ (P($x_1$) ∧ P($x_2$) ∧ ... ∧ P($x_n$))
    and ( ∃ x∈D, P(x)) ≡ (P($x_1$) ∨ P($x_2$) ∨ ... ∨ P($x_n$))


* **Vacuous truth of a universal statement** "For all balls on the table x, if x
  is in this (empty) bowl, x is blue"
* **Universal conditional statements**
* Meaning of *necessary*, *sufficient*, and *only if* in relation to quantified
    statements

(∀ x∈D, P(x) is sufficient for Q(x))  ≡  (∀ x∈D, P(x) ---> Q(x))
(∀ x∈D, P(x) is necessary for Q(x))  ≡  (∀ x∈D, Q(x) ---> P(x))
(∀ x∈D, P(x) only if Q(x))  ≡  (∀ x∈D, P(x) ---> Q(x))

* Logical equivalence of quantified statements: identical truth values, no
    matter what predicates are substituted for the predicate symbols, and no
    matter what sets are used for the domains of the predicate values.  For
    example both ~(∀ x∈D, P(x)) and (∃ x∈D, ~P(x)) have the same truth values,
    no matter what predicate P is, or what set D is.
* Negation of an Existential Statement: ~(∃ x∈D, P(x)) ≡ (∀ x∈D, ~P(x))
* Negation of a Universal Conditional:
        ~(∀ x∈D, P(x) --> Q(x))) ≡ (∃ x∈D, P(x) ∧ ~Q(x))


* Variants of Universal Conditional Statments

(i)   Statement: (∀ x∈D, P(x) --> Q(x)))
(ii)  Contrapositive: (∀ x∈D, ~Q(x) --> ~P(x)))
          (This is logically equivalent to i)
(iii) Converse: (∀ x∈D, Q(x) --> P(x)))
          (This is NOT logically equivalent to i)
(iv)  Inverse: (∀ x∈D, ~P(x) --> ~Q(x)))
          (This is logically equivalent to iii
              - it is the contrapositive to iii)

Sample illustrative problems for section 3.2

3.2.3a: Write a formal negation for (∀ fish x, x has gills.)

3.2.3c: Write a formal negation for
        (∃ a movie m such that m is over 6 hours long.)

3.2.16: Write a negation for (∀ real numbers x, if x2 >= 1 then x >0.)

3.2.22: Write a negation for (If the square of an integer is odd, then the
    integer is odd.)

3.2.32: Write the converse, inverse, and contrapositive of this statement:
      (If the square of an integer is odd, then the integer is odd.)
      Indicate which among the four statements is true, and which is false.
      Give counter examples for the ones that are false.

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**Day Seven: Wed, Sep 16, 2015**
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  * **Take Roll**
  * **Handback HWs??**
  * **Collect HW #5 on Section 3.1 due today**
  * **The next homework due is on section 3.2, due Monday, Sept 21**

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  * The first quiz is tentatively scheduled for Wednesday, Sept 30

Topics in Section 3.3 (Statements with Multiple Quantifiers)

* Truth of a ∀ ∃ Statement (for example, in a "Tarski World")
* Truth of a ∃ ∀ Statement (for example, in a "Tarski World")
* Key idea for the two above - imagine making "choices" in the order the
   quantifiers are given.
* Interpreting Multiply-Quantified Statements
Translating from Informal to Formal Language

Negations of Multiply-Quantified Statements:
    Example #1
       ~( ∀ x∈D, ∃ y∈E such that P(x,y) )
    ≡   ( ∃ x∈D, ~(∃ y∈E such that P(x,y)) )
    ≡   ( ∃ x∈D, ∀ y∈E, ~P(x,y) )
    Example #2
       ~( ∃ x∈D, ∀ y∈E, P(x,y) )
    ≡   ( ∀ x∈D, ~(∀ y∈E P(x,y)) )
    ≡   ( ∀ x∈D, ∃ y∈E, ~P(x,y) )
* Order of Quantifiers
      - If you interchange ∀ and ∃, usually it changes the meaning
           Example: ∀ people x, ∃ a person y such that x loves y
* Formal Logical Notation

Go over the Tarski World Examples 3.3.1, 3.3.2, 3.3.9
Do a&b of Example 3.3.10: Formalizing Statements in a Tarski World

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**Day Eight: Mon, Sep 21, 2015**
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  * Take Roll
  * Handback HW#3
  * Collect HW #6 on Section 3.2, due today
  * The next homework due is on section 3.3, due Wednesday, Sept 23
  * The first quiz is tentatively scheduled for Wednesday, Sept 30

Sample illustrative problems for section 3.3

3.3.9a: Let D = E = {-2, -1, 0, 1, 2} Explain why the following statement is
   true: ∀ x∈D, ∃ y∈E such that x+y=0.

3.3.15: (a) Rewrite the statement in English, without using the symbols ∀ or ∃
   or variables and expressing your answer as simply as possible, and (b)
   write a negation for the statement: ∀ odd integers n, ∃ an integer k such
   that n=2k+1.

3.3.34 (a) Rewrite the statement formally using quantifiers and variables,
   and (b) write a negation for the statement: Somebody loves everybody

3.3.37 (a) Rewrite the statement formally using quantifiers and variables,
   and (b) write a negation for the statement: Any even integer equals twice
   some integer.

**Topics in Section 3.4 (Arguments with Quantified Statements)**

* The rule of universal instantiation: If some property is true of everything
   in a set, it is true of any particular thing in the set.


* Universal Modus Ponens:
        ∀ x, If P(x) then Q(x);
        P(a) for a particular a,
        ∴ Q(a)

* Universal Modus Tollens:
        ∀ x, If P(x) then Q(x);
        ~Q(a) for a particular a,
        ∴ ~P(a)

* Validity of Arguments with Quantified Statements

* The Quantified Form of the Converse Error
        ∀ x, If P(x) then Q(x);
        Q(a) for a particular a,
        ∴ P(a) <--- invalid conclusion

* The Quantified Form of the Inverse Error
        ∀ x, If P(x) then Q(x);
        ~P(a) for a particular a,
        ∴ ~Q(a) <--- invalid conclusion

* Universal Transitivity
         ∀ x, P(x) --> Q(x)
         ∀ x, Q(x) --> R(x)
      ∴ ∀ x, P(x) --> R(x)

**Sample illustrative problems for section 3.4**

3.4.7: The argument may be valid by universal modus ponens or universal modus
    tollens.  It may be invalid and exhibit the converse or inverse error.
    State whether valid or invalid and justify your answer:

  All healthy people eat an apple a day
  Keisha eats an apple a day
∴ Keisha is a healthy person

3.4.8: The argument may be valid by universal modus ponens or universal modus
    tollens.  It may be invalid and exhibit the converse or inverse error.
    State whether valid or invalid and justify your answer:

  All freshmen must take writing
  Caroline is a freshman
∴ Caroline must take writing

* 3.4.23: Indicate whether the argument is valid or invalid.  Support your
    answer by drawing diagrams.

  All teachers occasionally make mistakes.
  No gods ever make mistakes.
∴ No teachers are gods.

* 3.4.24: Indicate whether the argument is valid or invalid.  Support your
    answer by drawing diagrams.

  No vegetarians eat meat.
  All vegans are vegetarian.
∴ No vegans eat meat.


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**Day Nine: Wed, Sep 23, 2015**
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 * **Take Roll**
 * **Collect HW #7 on Section 3.3, due today (W 2015/09/23)**
 * **The next homework due is #8 on section 3.4, due Monday, Sept 28**
 * **The first quiz is tentatively scheduled for Wednesday, Sept 30**
 * **Show students how to access the review guides for each chapter**

**Chapter Four (Elementary Number Theory and Methods of Proof)**

**Topics in Section 4.1 (Proof and Counterexample I: Introduction)**

* Definitions of even and odd integers (2k or 2k+1)
* Definition of a prime number: (p>1 in Z st if n,m in $Z^+$ and nm=p, then n=p
    or m=p)

* Definition of a composite number: (c>1 in Z, and c=nm for integers n>1 and
  m>1 )
* Constructive Proofs of Existential Statements (to show ∃ x∈D s.t. Q(x), one
  can either find an x that makes Q(x) true, or give a set of directions for
  finding an x that makes Q(x) true.  Either way, that's a constructive
  proof.)
* Non-Constructive Proof of Existence: (a) show existence is guaranteed by
  some theorem, or (b) show that the assumption that there is no x that
  makes Q(x) true leads to a contradiction.

* Disproving Universal Statements by Counterexample (Show P(x) ---> Q(x) is
  false by providing a counter example c such that P(c) && ~Q(c) )
* The Method of Exhaustion (show P(x) ---> Q(x) by individually checking each
  x such that P(x) is true.)
* Generalizing from the Generic Particular  (e.g. "Direct Proof": show P(x)
  ---> Q(x) by assuming x is some generic element of the domain, and basing
  the demonstration only on that)
* Existential Instantiation - if you have established that something exists,
  you can give it a name in your logical arguments, so long as you don't
  give it a name that is already being used for something else. (Example: if
  we know that m is an even number, then we know it is twice some integer,
  so we can give that integer the name k, and write m = 2k.)

**Sample illustrative problems for section 4.1**

8: Prove there is a real number x>1 such that $2^x > x^{10}$.

The Homework problem (#10) is to prove that there is an integer n such that
$2n^2 - 5n + 2$ is prime.  Hint: $2n^2 - 5n + 2$ can be factored as (n-2)*(2n-1)

Prove the statement.  Use only the definitions of the terms and the
  Assumptions listed on page 146, not any previously established properties
  of odd and even integers.  Follow the directions in this section for
  writing proofs of universal statements.

30: For all integers m, if m is even then 3m+5 is odd.

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**Day Ten: Mon, Sep 28, 2015**
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  * **Take Roll**
  * **Handback HW#??**
  * **Collect HW #8 on Section 3.4, due today**
  * **The next homework due is on section 4.1, due Wednesday, Sept 30**
  * **The first quiz is scheduled for Monday, Oct 05, and it will cover these**
    **sections:  1.1-1.3, 2.1-2.3, and 3.1-3.4**

**Math 2300, Section #1; 14:00-15:15 MW P-101; Daily Notes  Fall, 2015**

Problem area to review:
What is the logical negation of p ---> q ?
(example: if Bob is a cat, then Bob is a mammal)

Is the negation (p ---> ~q)?  NO

p ---> q means "it is not true that (p is true and q is false)"
(in the example:
  "it is not true that (Bob is a cat and Bob is not a mammal)" )

Consider Truth Table, and discover
p ---> q is logically equivalent to ~p ∨ q
      (e.g. Bob is not a cat, or Bob is a mammal)

Therefore the logical negation of p ---> q is (p ∧ ~q)
    (Bob is a cat AND Bob is not a mammal)

Why isn't (p ---> ~q) the equivalent to the logical negation of (p ---> q)?
 Because (p ---> ~q)
    means "it is not the case that (p is true and q is true)"
(e.g. it is not the case that (Bob is a cat and Bob is a mammal)
Which is logically equivalent to (~p ∨ ~q)
(Bob is not a cat OR Bob is not a mammal)

35: Prove the statement is FALSE: There exists an integer m >= 3 such that
    $(m^2-1)$ is prime.

39: Find the mistake in the "proof":
    Theorem: The difference between any odd integer and any even integer is
    odd.
"Proof: Suppose n is any odd integer, and m is any even integer.  By
    definition of odd, n=2k+1, where k is an integer, and by definition of
    even, m=2k, where k is an integer.  Then n-m = (2k+1)-2k = 1.  But 1 is
    odd.  Therefore, the difference between any odd integer and any even
    integer is odd."

55: Determine whether the statement is true or false.  Justify your answer
    with a proof or counter-example, as appropriate.  Use only the definitions
    of the terms and Assumptions listed on page 146, not any previously
    established properties.

Every positive integer can be expressed as a sum of three or fewer perfect
    squares.

50: Determine whether the statement is true or false.  Justify your answer
    with a proof or counter-example, as appropriate.  Use only the definitions
    of the terms and Assumptions listed on page 146, not any previously
    established properties.

For all integers n and m, if n−m is even then $n^3-m^3$ is even.

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**Day Eleven: Wed, Sep 30, 2015**
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  * **Take Roll**
  * **HW #8 (section 3.4) is not ready, so the quiz on Monday will cover only up
    to section 3.3**
  * **Collect HW #9 on Section 4.1, due today**
  * **The next homework due is on section 4.2, due Wednesday, Oct 7**
  * **The first quiz is scheduled for Monday, Oct 05, and it will cover these
    sections:  1.1-1.3, 2.1-2.3, and ~~3.1-3.4~~ 3.1-3.3**

**Topics in Section 4.2 (Proof and Counterexample II: Rational Numbers)**

* Rational Numbers (Q): Q = { r∈R │ r = a/b, for integers a,b, with b≠0}
  They are called rational because they are ratios of integers

* More on Generalizing from the Generic Particular - seen as being prepared
   to meet the challenge of showing P(x) is true for any particular value of
   x that is offered by an 'adversary'.

* Deriving New Mathematics from Old - Once you have proved something, you can
   use it in proofs of new things.

* A Corollary is a statement whose truth can be immediately deduced from a
   theorem that has already been proved.

**Sample illustrative problems for section 4.2**

* H16: The quotient of any two rational numbers is rational.
* 39: Find the flaw in the proof that the sum of two rational numbers is
   rational:
      "PROOF: Suppose r and s are rational numbers.  If r+s is rational then
   by definition of rational r+s = a/b where a,b are integers and b≠0.  Also,
   since r and s are rational, r=i/j, s=m/n for integers i,j,m,n with j≠0 and
   n≠0.  It follows that r+s = (i/j) + (m/n) = (a/b), which is a quotient of
   two integers, with a non-zero denominator.  Hence it is a rational number.
   This was what was to be shown."

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**Day Twelve: Mon, Oct 05, 2015**
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  * **Give Quiz**

**Math 2300, Section #1; 14:00-15:15 MW P-101; Daily Notes  Fall, 2015**

  * **Take Roll**
  * **Hand back HW #8 (section 3.4)**
  * **HW #9 ??**
  * **The next homework due is on section 4.2, due Wednesday, Oct 12**
  * **Report on status of the first quiz**

**Topics in Section 4.3 (Direct Proof and Counterexample III: Divisibility)**

* When $n \in Z$, and $d \in Z$ with $d \neq 0$, $d|n$ means "d divides n," which means that $n=dm$,
  where $m \in Z$.  For example $2|10$ because $10=2*5$.  We also express this idea by
  saying "n is a multiple of d," "d is a factor of n," and "d is a divisor
  of n."

* Divisors of zero: If $d \in Z$ with $d \neq 0$, we say d is a "divisor of 0" because it
  is true that $0=d*0$.  (Every non-zero integer is divisor of 0.)  For
  example $0=42*0$, and $42 \neq 0$, so 42 is a divisor of 0.

* However 0 is not a divisor of anything.

* Theorem 4.3.1 – A Positive Divisor of a Positive Integer: For all integers
  m, n, if m and n are positive and $m|n$, then $m \leq n$.
(Idea of the proof: $n=mk$, $k \in Z$. k must be $\geq 0$. $\therefore$ $k \geq 1$, so $mk \geq m$, i.e. $n \geq m$, qed)

* Theorem 4.3.2 – Divisors of 1: The only divisors of 1 are 1 and –1.
(Idea of the proof: $1*1=1$ and $(-1)*(-1)=1$ shows that 1 and –1 are divisors of
  1.  If m is any divisor of 1, then $1=mk$, $m \in Z$ and $k \in Z$. If m and k are
  positive then by Theorem 4.3.1, $1 \geq m$. The only positive integer $\leq 1$ is 1, so
  that means m is 1.  If m and k are not both positive, they must both be
  negative.  In that case, $1=(-m)(-k)$, where both –m and -k are positive.
  Again by Theorem 4.3.1, we can conclude that -m is 1, in other words $m=-1$.

* Theorem 4.3.3 – Transitivity of Divisibility: $\forall$ $k \in Z$, $m \in Z$, $n \in Z$,
        if $k|m$ and $m|n$, then $k|n$
(Idea of proof: $m=kd$, $n=mh$, where $d \in Z$ and $h \in Z$. $\therefore$ $n=mh=(kd)h=k(dh)$. $dh \in Z$, so we
  have shown that $k|n$. qed)

* Theorem 4.3.4 – Divisibility by a Prime: Any integer $n>1$ is divisible by a
  prime number.
(Idea of proof: If n is prime, then it is divisible by a prime number –
  itself.  If n is not prime then $n=r_0 s_0$ where $r_0$ and $s_0$ are both integers
  "properly" between 1 and n.  $r_0$ divides n.  If $r_0$ is prime, then n is
  divisible by a prime.  If $r_0$ is not prime then $r_0=r_1 s_1$, where $r_1$ and $s_1$ are
  both integers "properly" between 1 and $r_0$. By the transitivity of
  divisibility, $r_1|n$. Also $1<r_1<r_0<n$.  If $r_1$ is prime, then n is divisible by

a prime. If not, $r_1$ can be factored as $r_1=r_2 s_2$, where $r_2$ and $s_2$ are both integers "properly" between 1 and $r_1$, $r_1|n$ (by transitivity of divisibility), and $1<r_2<r_1<r_0<n$.  If $r_2$ is prime, then n is divisible by a prime.  This process of finding integers $r_i$ has to stop with finding an $r_i$ that is prime eventually, because if it did not, there would be an infinite <u>descending</u> sequence of integers greater than 1: $n>r_0>r_1>r_2>r_3> ...$ > 1.  This is a contradiction because, whatever positive integer n is, there are only finitely many integers between n and 1. Since the process stops with an $r_i$ that is prime, and since that $r_i$ divides n (by transitivity of divisibility), we have proved that n is divisible by a prime. qed)

* Theorem 4.3.5 – Unique Factorization of Integers Theorem (The Fundamental Theorem of Arithmetic): Given any integer n>1, n can be expressed as the product of a list of prime numbers.  (In this kind of list, the same prime is allowed to appear multiple times.) Except for different possible orderings, there is only one such list for each n.

  Example: 24 = 2*2*2*3, and there is no other way to write 24 as a product of primes, except for shuffling the factors around like this: 24=2*2*3*2.

**Sample illustrative problems for section 4.3**

* 15: Prove directly from the definition of divisibility:
    For all integers a, b, and c, if a|b and a|c, then a|(b+c)
* H19: Determine whether the statement is true or false,.  If true, prove directly from the definitions.  If false, give a counter example.

  For all integers a, b, and c, if a divides b then a divides bc.

* 20: Determine whether the statement is true or false,.  If true, prove directly from the definitions.  If false, give a counter example.

  The sum of any three consecutive integers is divisible by 3.  (Integers m<n are consecutive if and only if n=m+1.)

* 31: Determine whether the statement is true or false,.  If true, prove directly from the definitions.  If false, give a counter example.

  For all integers a and b, if a|10b, then a|10 or a|b.

**Math 2300, Section #1; 14:00-15:15 MW P-101; Daily Notes  Fall, 2015**

  * **Take Roll**
  * **Hand back HW #9 on Section 4.1. (Figure out whose paper has no name.)**
  * **Homework #10 on section 4.2 is due today.**
  * **Homework #11 on section 4.3 is assigned today. It is due on Wednesday,
     October 14.**
  * **Report on status of the first quiz**

**Topics in Section 4.4 (Proof and Counterexample IV: Division into Cases and
    the Quotient Remainder Theorem)**

* The basic idea of dividing an integer j by another integer k is to
    'represent j as some groups of size k'.

* For example, the idea of dividing 13 by 5 is to express 13 as two groups of
    five, with three left over.

            11111 11111 111  = 13 = 2*5+3

* Theorem 4.41 – The Quotient Remainder Theorem: Given any integer n and
    positive integer d, there exist unique integers q and r such that

        n = (d * q) + r  and 0 <= r < d

Example:
 53 = 3 * 17 + 2;  So here q=17 and r=2 (0 <= r < d)
-53 = 3 * (-17) – 2 = 3 * (-18) + 1; So here q=(-18) and r=1 (0 <= r < d)
    (When d is a factor of n, n=d*k for some integer k and the remainder r=0.
     In this case, -n=d*(-k) and the remainder is also zero.  When n=d*k+r,
   with 0 < r < d (in other words r≠0) then the quotient remainder numbers for
    -n are:

            -n = d*(-k-1) + (d-r)

The example above of 53=3*17+2 and -53 =3*(-18)+1 illustrate the idea.

When n>0 and d>0, and n=dk+r (0<=r<d), k=n/d (k=n div d) and r=n%d (r = n mod
    d), where / and % are the C++ (or Java) operators.

* We can use Theorem 4.4.1 to prove that every integer is either even or odd.
(Idea of Proof: n=2q+r, where 0<=r<2.  r must be 0 or 1.  Therefore n is
    either even or odd. The uniqueness of q and r implies that n cannot be
    both even and odd.)

**Math 2300, Section #1; 14:00-15:15 MW P-101; Daily Notes  Fall, 2015**

* Method of Proof by Division into Cases
  To prove a statement of the form "If $A_1$, or $A_2$, or ... or $A_n$, then C," prove
  all of "If $A_1$ then C, if $A_2$ then C, ... and if $A_n$ then C." The process
  shows that C is true regardless of which of  $A_1$, $A_2$, ..., $A_n$ happens to be
  the case.

An example of this is a proof that two consecutive integers have opposite
  parity.  It's convenient to divide the proof into the two cases where the
  first integer is even, and the case where the first integer is odd.

* Theorem 4.4.6 - The Triangle Inequality: Let x, y be any two real numbers.
  $|x+y|<=|x|+|y|$.

* Lemma 4.4.4: For any real number r, $-|r|<=r<=|r|$.
  Proof: if r>=0 then the inequalities just say that $-r<=r<r$, which is
  obviously true.
        On the other hand if r<0, then the inequalities just say that
  $r<=r<=-r$, which is also obviously true.

(Idea of the proof of the triangle inequality:
      From the lemma, we know that $x<=|x|$ and $y<=|y|$ are true.
      Therefore $x+y <= |x|+|y|$.
      Case 1: x+y >= 0.  In this case $|x+y| = x+y$.  Since $x+y <= |x|+|y|$,
            we can conclude $|x+y| <= |x|+|y|$ in case 1.
      Case 2: x+y < 0. In this case $|x+y| = -(x+y) = (-x) + (-y)$.
            We know from the lemma that $(-x) <= |-x| = |x|$,
            and $(-y) <= |-y| = |y|$.
            Therefore $(-x) + (-y) <= |x|+|y|$.
            Since, in this case $|x+y| = (-x) + (-y)$,
            we can also conclude that $|x+y| <= |x|+|y|$ in case 2.


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**Sample illustrative problems for section 4.4**


* Problem 4.4.23: Prove that for all integers n, if n mod 5 = 3, then $n^2$ mod 5
  = 4.
  (In other words if the remainder upon division of n by 5 is 3, then the
  remainder upon division of $n^2$ by 5 is 4.)

* Problem 4.4.51: If m, n, a, b, and d are integers, d>0 and m mod d = a and
  n mod d = b, is (m+n) mod d = (a+b)?  Is (m+n) mod d = (a+b) mod d?  Prove
  your answers.

**Math 2300, Section #1; 14:00-15:15 MW P-101; Daily Notes  Fall, 2015**

  * Take Roll
  * Hand back HW #9 on Section 4.1. (Figure out whose paper has no name.)
  * Homework #11 on section 4.3 is due today.
  * Homework #12 on section 4.4 is assigned today. It is due on Monday,
     October 19.
  * Hand back Quiz #1 - Your percentage is on the second page
                       - Go over the answers -
     Scores, when rounded mean:
        91 -100:  A;
        81 - 90:  B;
        71 - 80   C;
        61 - 70   D;
         0 - 60   F;

  * Take Roll
  * Hand back?? Homework #11 on section 4.3, due Wednesday, Oct 14.
  * Collect Homework #12 on section 4.4, due today, Monday, October 19.
  * Possibly assign Homework #13
  * Hand back remaining Quiz #1


      - Your percentage is on the second page


     Scores, when rounded mean:
        91 -100:  A;
        81 - 90:  B;
        71 - 80   C;
        61 - 70   D;
         0 - 60   F;

  Revisit the proof of the UNIQUENESS of the quotient and the remainder in the
     Quotient-Remainder Theorem

  Suppose $n = dh + r = dk + s$, where $n,d,h,k,r,s$ are integers, $d>0$, $0 \le r < d$, and
     $0 \le s < d$.

Case 1 - $h \ge k$: (See proof in class)

Case 2 - $k \ge h$: (See proof in class)

**Math 2300, Section #1; 14:00-15:15 MW P-101; Daily Notes  Fall, 2015**

**Sample illustrative problems for section 4.4**

* Problem 4.4.23: Prove that for all integers n, if n mod 5 = 3, then $n^2$ mod 5
  = 4.
  (In other words if the remainder upon division of n by 5 is 3, then the
  remainder upon division of $n^2$ by 5 is 4.)

* Problem 4.4.51: If m, n, a, b, and d are integers, d>0 and m mod d = a and
  n mod d = b, is (m+n) mod d = (a+b)?  Is (m+n) mod d = (a+b) mod d?  Prove
  your answers.

**Topics in Section 4.6 (Indirect Argument: Contradiction and Contraposition)**

* The Method of 'Proof By Contradiction' (this method uses the 'rule of the
  excluded middle' – the idea that if a statement is NOT FALSE, then it must
  be TRUE, because there's presumably nothing 'in between' TRUE and FALSE.)
  1. If you want to prove S by contradiction, begin by making ~S, the
     logical negation of S, a PREMISE.
  2. Make logical deductions that lead to a contradiction – something that
     is known to be FALSE.
  3. Observe that ~S must be FALSE, because to assume it is true leads to a
     FALSE conclusion.  Observe that S must therefore be TRUE. (~S is FALSE
     means "S is TRUE")

Example: Let {$p_1$, $p_2$, ..., $p_n$} be a finite list of n>=1 prime numbers, where
         is an integer.
         Let q = $p_1$*$p_2$*...*$p_n$ + 1,
             in other words q is one more than the product
             of the list of primes.

         Prove this statement S:
             "None of the primes in the list is a divisor of q."

         Proof:  Assume ~S is true.  In other words assume that there is a
                 prime p in the list such that p|q, i.e. q=pk, where k is
                 an integer.

                 Note also that p divides $p_1$*$p_2$*...*$p_n$
                 – in other words $p_1$*$p_2$*...*$p_n$ = ph where h is the product
                 of all the primes in the list that are not equal to p.
                 (If p is the only prime in the list, then h=1.)

                 Since q = $p_1$*$p_2$*...*$p_n$ + 1,
                     1 = q-$p_1$*$p_2$*...*$p_n$ = pk-ph = p(k-h).

         This shows that 1 is a multiple of p.

         p is positive, by definition of prime,

```
        the product p(k-h)= 1 is positive
        Therefore (k-h) is positive
           (it can't be negative or zero,
            because 1 is not negative or zero.)

        p>1 by definition of a prime.

   Since p>1 and (k-h) is positive,
   we can multiply p>1 and get (k-h)p > (k-h).

 Since (k-h) is positive, and (k-h) is an integer, (k-h) ≥ 1, and so putting
   together some things we've proved:

        1=p(k-h)>(k-h)≥ 1, by transitivity, 1>1, which is a contradiction.
```

Since assuming ~S leads to the contradiction above, ~S must be FALSE.
In other words, S must be TRUE: $q = p_1*p_2*...*p_n + 1$, is not divisible by any
    of the primes $\{p_1, p_2, ..., p_n\}$. qed

**Sample illustrative problems for section 4.6**

* Carefully formulate the negations of the statement.  Then prove the
    statement by contradiction.

  4.6.11: *S = "The product of any nonzero rational number and any irrational*
  *number is irrational."*

The negation ~S: "There exists a rational number $q\neq0$, and an irrational number
    s, such that the product qs is rational."


Proof of S:

Assume that ~S is true.  q = m/n for integers m and n with m≠0 and n≠0. (We
    know m≠0 because q≠0 is given.) Since qs is rational,
     we have (m/n)s = (i/j) where i and j are integers and j≠0.
    It follows that (n/m)(m/n)s = (n/m)(i/j), which simplifies to
        s = (ni/mj).  We note that ni and mj are integers, and mj≠0
            can be concluded because m≠0 and n≠0.
    Thus s is rational, by definition of rational.  This contradicts the
    assumption of ~S, which included the assumption that s was irrational.
Therefore ~S must be FALSE, so S must be TRUE.  qed.

* Prove by contraposition:

  4.6.19: S= "If a product of two positive real numbers is greater than 100,
    then at least one of the numbers is greater than 10."

Proof: We attempt to prove the contrapositive of S: If two positive real
    numbers are both ≤ 10, then their product is ≤ 100.  (This is logically
    equivalent to S.)
So suppose 0<a≤10 and 0<b≤10.  Since b>0, multiplying through a≤10 yields

                ab≤10b

Also multiplying through b≤10 by 10 yields: 10b≤100.
Using transitivity, we get ab≤100. qed.

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**Day Seventeen: Wed, Oct 21, 2015**
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**Topics in Section 5.1 (Sequences)**

* Definition of a sequence
    A function that assigns a real number to each element of a subset S of the
    integers.  The set S is either of the form S = {i∈Z|i≥n}, where n is some
    integer, or of the form S = {i∈Z|m≥i≥n}, where m and n are integers, with
    m≥n.

        Example: $a_i$ = (−1)$^i$/(i!), i≥1 = −1, 1/2, −1/6, 1/24, −1/120, ...

* Definition of term, subscript, index, initial term, final term, subscript
    notation, meaning of ellipses, infinite sequence, explicit formula,
    general formula (formulae for a general term of the sequence, expressed as
    a function of the subscript)

* An alternating sequence ... See the previous example.

* Finding an explicit formula to fit given initial terms
* Summation Notation (example below)

$$\sum_{k=3}^{10} 2k + 1$$

* expanded form, index of a summation, lower limit, upper limit
* Computing summations
* Summation terms given by a formula
* Changing between summation form and expanded form
* Separating off terms
* Telescoping sums
* Product notation (example below)

$$4! = \prod_{k=1}^{4} k = 1 \times 2 \times 3 \times 4 = 24$$

* Theorem 5.1.1: Properties of Summations and Products

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**Theorem 5.1.1**

If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and $c$ is any real number, then the following equations hold for any integer $n \geq m$:

1. $\displaystyle\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$

2. $\displaystyle c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$    generalized distributive law

3. $\displaystyle\left( \prod_{k=m}^{n} a_k \right) \cdot \left( \prod_{k=m}^{n} b_k \right) = \prod_{k=m}^{n} (a_k \cdot b_k).$

</div>

* Change of variable - example

$$\sum_{k=0}^{6} \frac{1}{k+1} = \sum_{j=1}^{7} \frac{1}{j}.$$

* Factorial and "n choose r" notation
* Sequences in Computer Programming
* Dummy variable in a loop
* Applications

**Sample illustrative problems for section 5.1**

* Write the first four terms of the sequences defined by the formula

5.1.3: $c_i = [(-1)^i]/3^i$ for all integers $i \geq 0$

* Write the first four terms of the sequences defined by the formula

5.1.5  $e_n = \lfloor n/2 \rfloor \cdot 2$  for all integers $n \geq 0$

* Find an explicit formula for the sequence

5.1.12: 1/4, 2/9, 3/16, 4/25, 5/36, 6/49

* Write using summation or product notation

5.1.44: (2/(3*4))-(3/(4*5))+(4/(5*6))-(5/(6*7))+(6/(7*8))

* Write using summation or product notation

5.1.50: 1/2! + 2/3! + 3/4! + ... + n/(n+1)!

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**Day Eighteen: Mon, Oct 26, 2015**
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  * **Take Roll**
  * **Hand back left-over homeworks.**
  * **Collect Homework #14 on section 5.1 due today, Monday, October 26.**
  * **Possibly assign Homework #15**

**Topics in Section 5.2 (Mathematical Induction I)**

* The Principle of Mathematical Induction
  Let P(n) be a statement that is defined for integers n.
  Let c be a fixed integer.  Suppose the following two statements are true:

  1. P(c) is true.
  2. For all integers k≥c, if P(k) is true, then P(k+1) is true.

If so, then P(n) is true for all integers n≥c

* A principle logically equivalent to the principle of mathematical
  induction:

    Suppose S is any set of integers satisfying

      1. c ∈ S
      2. for all k≥c if k ∈ S then k+1 ∈ S

Then S must contain every integer greater than or equal to c.

* (S is the set of all integers for which the theorem is true, according to
  the principle of mathematical induction)

* The basic step - proving P(c)

* The inductive hypothesis - the assumption that P(k) is true, ∃ k ∈ Z, k≥c

* The inductive step - proving that P(k) --> P(k+1)

* A *closed form* is a formula for the value of a sum that does not contain an
  ellipsis or a summation sign.

    Example: n(n+1)/2 is a closed form for 1 + 2 + ... + n.

**Sample illustrative problems for section 5.2**

* 5.2.10 Prove the statement by mathematical induction:

$$1^2 + 2^2 + ... + n^2 = n(n+1)(2n+1)/6 \text{ for integers } n \geq 1$$

* 5.2.13 Prove the statement by mathematical induction:

The sum from i=1 to n-1 of i(i+1) equals n(n-1)(n+1)/3 for integers n≥3


* 5.2.20 Use the formula for the sum of the first n integers and/or the
   formula for the sum of a geometric sequence to evaluate the sum, or to
   write it in closed form:

$$4 + 8 + 12 + 16 + ... + 200$$

* 5.2.22 Use the formula for the sum of the first n integers and/or the
   formula for the sum of a geometric sequence to evaluate the sum, or to
   write it in closed form:

$$3 + 4 + 5 + 6 + ... + 1000$$

* 5.2.27 Use the formula for the sum of the first n integers and/or the
   formula for the sum of a geometric sequence to evaluate the sum, or to
   write it in closed form:

$$5^3 + 5^4 + 5^5 + ... + 5^k, \text{ where k is any integer with } k \geq 3.$$

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**Day Nineteen: Wed, Oct 28, 2015**
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**Topics in Section 5.3 (Mathematical Induction II)**

* Proving divisiblity of $2^{2n} - 1$ by 3

* Proving the inequality $2n+1 < 2^n$, for n≥3

* Proving a property of a sequence, e.g. when $a_1$ = 2, and $a_k = 5a_{k-1}$
  The problem is to show that $a_n = 2(5^{n-1})$ for n≥1.

* "A Problem with Trominoes"

**Sample illustrative problems for section 5.3**

* 5.3.6
* 5.3.10
* 5.3.21

**Math 2300, Section #1; 14:00-15:15 MW P-101; Daily Notes  Fall, 2015**

**Topics in Section 5.4 (Strong Mathematical Induction and the Well-Ordering**
   **Principle for the Integers)**

* Principle of Strong Mathematical Induction

   Let P(n) be a property that is defined for integers n, and let a and b be
   fixed integers with a≤b.  Suppose the following two statements are true:

   1) P(a), P(a+1), ... and P(b) are all true (basis step)
   2) For any integer k≥b, if P(i) is true for all integers i from a through
   k, then P(k+1) is true. (inductive step)

   Then the statement

         for all integers n≥a, P(n)

   is true.  (The supposition that P(i) is true for all integers i from a
   through k is called the inductive hypothesis.  Another way to state the
   inductive hypothesis is to say that P(a), P(a+1), ... P(k) are all true.)

* Actually the Principle of Strong Mathematical Induction is not really any
   'stronger' than the Principle of Mathematical Induction - anything that
   can be proved with one can also be proved with the other.  However
   sometimes one or the other is more convenient for someone constructing a
   proof.

* The Principle of Strong Mathematical Induction is also known as the second
   principle of induction, the second principle of finite induction, and the
   principle of complete induction.

* Using the Principle of Strong Mathematical Induction, we can conveniently
   prove that every integer greater than 1 is divisible by a prime.

* Another example is a proof of a property of a sequence

* Another example is proof that a multiplication of k factors always requires
   k-1 multiplications, regardless of how the factors are associated.

* Another example: Existence and uniqueness of binary integer representations

* The Well Ordering Principle is equivalent to both the ordinary and the
   strong principles of mathematical induction.

The Well Ordering Principle:

  Every non-empty set of positive integers contains a least element.

* Equivalently, every non-empty set of integers that is bounded below
 contains a least element.

* Application: Proof of existence part of the Quotient Remainder Theorem.

  (S = set of all non-negative integers of the form n-dk,
   where k is an integer)

Here is a proof of the existence part of the Quotient Remainder Theorem that
 uses only ordinary mathematical induction.

**Theorem: $\forall$ n $\in$ Z, $\forall$ d, 1≤d∈Z, $\exists$ q∈Z, $\exists$ r∈Z, such that**
   **P(n) is true, where P(n) is this statement: n=dq+r, and 0≤r<d**

**Proof: Case #1, assume n≥0.  We prove case #1 by induction.**
**(1) 0 = d*0+0 establishes that P(0) is true with q=0, and r=0**
**(2) Suppose that P(k) is true, $\exists$ 0≤k∈Z.**
 **Then k = dh+s, where h∈Z, s∈Z, and 0≤s<d.**
 **It follows that k+1=dh+(s+1).  We know that 0<s+1≤d. If s+1<d,**
 **then k+1=dh+(s+1) establishes that P(k+1) is true.  If it is not true**
 **that s+1<d, then s+1=d.  In that case, this is true: k+1=dh+d=d(h+1),**
 **which shows that P(k+1) is true with q=h+1, and r=0.**
 **We have demonstrated that P(k) --> P(k+1)**
**It follows from the principle of mathematical induction that P(n) is true**
**for all integers n≥0.**

**Case #2: Assume n<0.  By the proof above, P(-n) is true.  Therefore**

 **-n = dp+t, where p∈Z, t∈Z, and 0≤t<d**

 **Therefore n = d(-p)-t, and (-d)<(-t)≤0**

 **If t=0, then this shows that P(n) is true with q=-p, and r=0.**
 **If t≠0, then (-d)<(-t)<0, and writing n this way**

 **n = d(-p-1)+(d-t),**

 **we observe that (-d)<(-t)<0 implies 0<(d-t)<d,**

 **which shows P(n) to be true with q=(-p-1) and r=(d-t).**

**This establishes that P(n) is true for integers n<0.  So cases #1 and #2**
 **combined prove that P(n) is true for all integers n, qed.**

* Application: Proof that a strictly decreasing sequence of non-negative
  integers is finite.

**Sample illustrative problems for section 5.4**

* 5.4.7:  $g(1)=3$, $g(2)=5$, $g(k)=3g(k-1)-2g(k-2)$, $\forall k$, $3\leq k \in Z$
          Prove: $g(n)=2^n+1$, $\forall n$, $1\leq n \in Z$

* 5.4.8: $h(0)=1$, $h(1)=2$, $h(2)=3$, $h(k)=h(k-1)+h(k-2)+h(k-3)$, $\forall k$, $3\leq k \in Z$
          Prove: (a) $h(n)\leq 3^n$, $\forall n$, $0\leq n \in Z$
                 (b) Suppose s is any real number such that $s^3 \geq s^2+s+1$
                     (Such a number s must be more than 1.83.)
                     Prove: $h(n)\leq s^n$, $\forall n$, $2\leq n \in Z$

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**Day Twenty-One: Wed, Nov 4, 2015**
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**Topics in Section 5.6 (Defining Sequences Recursively)**

* **A recurrence relation for a sequence $a_0$, $a_1$, $a_2$, ... is a formula that
  relates each term $a_k$ to certain of its predecessors $a_{k-1}$, $a_{k-2}$, ..., $a_{k-i}$,
  where i is an integer with $k-i\geq 0$.  The initial conditions for such a
  recurrence relation specify the values of $a_0$, $a_1$, $a_2$, ..., $a_{i-1}$, if i is a
  fixed integer, or $a_0$, $a_1$, ..., $a_m$, where m is an integer with $m\geq 0$, if i
  depends on k.**

* Computing Terms of a recursively defined sequence
  E.g. $f(1)=1$, $f(2)=1$, $f(n)=f(n-1)+f(n-2)$ ... calculate $f(4)$ and $f(5)$.

* Writing a recurrence relation in more than one way
  E.g. $f(1)=1$, $f(2)=1$, $f(n)=f(n-1)+f(n-2)$
       $h(0)=1$, $h(1)=1$, $h(k+1)=h(k)+h(k-1)$

* (Different) Sequences that satisfy the same recurrence relation
  E.g. The Fibonacci sequence satisfies $f(n) = f(n-1)+f(n-2)$ for $n\geq 2$
       The Lucas sequence satisfies the same relation.  However the initial
       conditions are different.
           Fibonacci: $f(0)=1$, $f(1)=1$
           Lucas:     $f(0)=1$, $f(1)=3$

* Showing that a sequence given by an Explicit Formula Satisfies a Certain
  Recurrence Relation
  E.g. Proof of the recursion relation $C(k) = C(k-1)(4k-2)/(k+1)$ for the
       Catalan numbers $[C(n) = (2n \text{ choose } n) * (1/(n+1))$

* Examples of Recursively Defined Sequences

* The *recursive paradigm* (aka the *recursive leap of faith*)

* The Tower of Hanoi
* The Fibonacci Numbers
* Compound Interest
* Recursive Definitions of Sum and Product

**Sample illustrative problems for section 5.6**

* 5.6.3: Find the first four terms
        c(k) = k(c(k-1))^2, k≥1,; c(0)=1

* 5.6.8: Find the first four terms
        v(k)=v(k-1)+v(k-2)+1, k≥3; v(1)=1, v(2)=3

* 5.6.13: t(n)=2+n, n≥0; Show that t(k)=2t(k-1)-t(k-2), k≥3


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**Day Twenty-Two: Mon, Nov 9, 2015**
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  * **Take Roll**
  * **Hand backs through 5.4 – HW#17**
  * **Collect Homework #18 on section 5.6, due today, Monday, November 9.**
  * **Possibly assign Homework #19**

**Topics in Section 5.7 (Solving Recurrence Relations by Iteration)**

* Given a sequence that satisfies a ***recurrence relation***, a ***solution*** to the
    sequence is an ***explicit formula*** for each member of the sequence.

    Example: The Fibonacci sequence
            Recurrence relation: $F_1=1$, $F_2=1$, $F_n=F_{n-1}+F_{n-2}$, $\forall n$, $2 \leq n \in \mathbb{Z}$
            Solution (explicit formula):

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$$

where:

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61803\,39887\cdots$$

* The *Iteration Method* of finding a solution: generate terms until you see a
  pattern, guess an explicit formula, and verify it.

Example: $a_0=1$, $a_n=a_{n-1}+2n+1$, $\forall n$, $1 \leq n \in Z$
Solution by iteration: $a_1=a_0+2(1)+1=4$, $a_2=a_1+2(2)+1=9$,
$\qquad\qquad\qquad a_3=a_2+2(3)+1=16$, $a_4=a_3+2(4)+1=25$

So it looks like $a_n=(n+1)^2$ is an explicit formula for $a_n$.

The next step is usually to attempt to prove the guessed formula is correct,
  typically with proof by induction.

* Arithmetic sequence: $\{a_0, a_1, a_2, ...\}$, $\exists d \in Z$, $a_n=a_{n-1}+d$, $\forall n$, $1 \leq n \in Z$
  $\qquad\qquad\qquad$ (solution: $a_n=a_0+dn$, $\forall n$, $0 \leq n \in Z$)

* Geometric sequence: $\{a_0, a_1, a_2, ...\}$, $\exists d \in Z$, $a_n=da_{n-1}$, $\forall n$, $1 \leq n \in Z$
  $\qquad\qquad\qquad$ (solution: $a_n=a_0d^n$, $\forall n$, $0 \leq n \in Z$)
* other formulas:
  Arithmetic Series: $1+2+3+...+n = n(n+1)/2$
  Geometric Series: $1+x+x^2+...+x^n = (x^{n+1}-1)/(x-1)$; $0 \neq x \in R$

**Sample illustrative problems for section 5.7**

* 5.7.1b: Find a formula for the expression $3+2+4+6+8+...+2n$; $\forall n$, $1 \leq n \in Z$

* 5.7.6: Use iteration to guess an explicit formula:
  $\qquad d_1=2$; $d_k=2d_{k-1}+3$, $\forall k$, $2 \leq k \in Z$

* 5.7.10: Use iteration to guess an explicit formula:
  $\qquad h_0=1$; $h_k=2^k - h_{k-1}$, $\forall k$, $1 \leq k \in Z$

**Math 2300, Section #1; 14:00-15:15 MW P-101; Daily Notes  Fall, 2015**

  * **Quiz #2**

  * **Take Roll**
  * **Hand backs through 5.6 - HW#18**
  * **Possibly assign Homework #20**

* 5.7.35: Use mathematical induction to verify the correctness of the formula
   derived in exercise 5.7.10.

**Topics in Section 5.8 (Second-Order Linear Homogeneous Recurrence Relations
   with Constant Coefficients)**

* A second-order linear homogeneous recurrence relation with constant
   coefficients has the form

$$x_n = Ax_{n-1} + Bx_{n-2}, \text{ for all n greater than some fixed integer,}$$
$$\text{where A and B} \neq 0 \text{ are fixed constant real numbers.}$$

It's "second-order" because the value farthest back in the sequence used to
   define $x_n$ is $x_{n-2}$.

It's "linear" because each term contains no more than one of $x_n$, $x_{n-1}$, or $x_{n-2}$:
   there are no second or higher powers of $x_n$, $x_{n-1}$, or $x_{n-2}$, and no products of
   two or more of $x_n$, $x_{n-1}$, or $x_{n-2}$.

It's "homogeneous" because every term has the same "degree" - the same total
   number of powers of any of $x_n$, $x_{n-1}$, or $x_{n-2}$.  (When a recurrence relation is
   linear, the only way it can fail to be homogeneous is if it has a constant
   term.)

It has "constant coefficients" because A and B are fixed constants that do
   not depend on n.

* If t is some number such that $\mathbf{t^2 = At + B}$, then $t^k = At^{k-1} + Bt^{k-2}$ for all integers
   $k \geq 2$, which can be proved just by multiplying through the first equation by
   t multiple times.

* The equation $t^2 = At + B$, and equivalently $\mathbf{t^2 - At + B = 0}$, is called the
   ***characteristic equation*** of the relation.  Lemma 5.8.1 says basically that
   the sequence $\{x_0 = 1, x_1 = s, x_2 = s^2, x_3 = s^3, x_4 = s^4, \dots\}$ satisfies the relation
   $x_n = Ax_{n-1} + Bx_{n-2}$ if and only if s is a root of the characteristic equation.

Example: Consider the recurrence relation $x_n = x_{n-1} + 2x_{n-2}$

The characteristic equation is $T^2 - T - 2 = 0$, which factors as $(T-2)(T+1) = 0$.  So
   the solutions are T=2 and T=-1.  By lemma 5.8.1, the sequences

$\{1, 2, 2^2, 2^3, \ldots\}$ and $\{1, -1, (-1)^2, (-1)^3, \ldots\}$ satisfy the recurrence, and are the
   ONLY sequences of powers that satisfy the recurrence.

* Any 'linear combination' of solutions to the recurrence $x_n = Ax_{n-1} + Bx_{n-2}$ is
   also a solution.

Example: We showed that these sequences $\{1, 2, 2^2, 2^3, \ldots\}$ and $\{1, -1, (-1)^2,$
   $(-1)^3, \ldots\}$ are solutions to the recurrence, $x_n = x_{n-1} + 2x_{n-2}$

So, choosing any two arbitrary constants C and D, we can verify that

   $\{C+D, \; 2C-D, \; 2^2D+(-1)^2C, \; 2^3D+(-1)^3C, \; \ldots\}$ is also a solution to the
   recurrence.

Proof: Suppose $x_n = Ax_{n-1} + Bx_{n-2}$ and $y_n = Ay_{n-1} + By_{n-2}$ for sequences

$\{x_n\}$ and $\{y_n\}$.  Then $Cx_n + Dy_n$
                  $= C(Ax_{n-1} + Bx_{n-2}) + D(Ay_{n-1} + By_{n-2})$ // by the assumption
                  $= A(Cx_{n-1} + Dy_{n-1}) + B(Ax_{n-2} + Dy_{n-2})$ // by algebra

The latter term shows that the recursion relation holds for $Cx_n + Dy_n$

* If you have specific values $z_0$ and $z_1$ that you want for the first two terms
   of the solution sequence, and if you have two sequences $\{x_n\}$ and $\{y_n\}$ that
   satisfy the recurrence relation, you may be able to solve the two
   equations $Cx_0 + Dy_0 = z_0$ and $Cx_1 + Dy_1 = z_1$ for the values of C and D that give
   $z_0$ and $z_1$ as the first two terms of a solution.

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**Day Twenty-Five: Mon, Nov 23, 2015**
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  * **Take Roll**
  * **Assign Homework #20**

* If s and t are two distinct solutions to the characteristic equation, then
   the sequences $\{1, s, s^2, s^3, s^4, \ldots\}$ and $\{1, t, t^2, t^3, t^4, \ldots\}$ are solutions to
   the recurrence, and the equations $C+D = z_0$ and $Cs+Dt = z_1$ can be solved to
   get any two desired numbers $z_0$ and $z_1$ for the first two terms of a solution
   sequence.  Since $s \neq t$, the equations are guaranteed to have a solution.

* Since any solution to the recurrence is completely determined by the first
   two values and the recurrence relation, the information in the previous
   bullet indicates that all solutions to the recurrence are of the form

$\{Cs^n+Dt^n\}$ when s and t are two distinct roots of the characteristic equation.

* Double root case: If the recurrence relation is $x_n=Ax_{n-1} + Bx_{n-2}$ and r is a double root of the characteristic equation $T^2-AT-B=0$, then $(T-r)^2$ is the characteristic equation, which is $T^2-2rT+r^2=0$.  Thus $A=2r$, $B=-r^2$ and

$$\mathbf{A(k-1)r^{k-1} + B(k-2)r^{k-2}} = Akr^{k-1} + Bkr^{k-2} - Ar^{k-1} - 2Br^{k-2}$$
$$= kr^{k-2}(Ar+B) - r^{k-2}(Ar+2B)$$

Because r is a root of $T^2-AT-B=0$, $Ar+B=r^2$ and because $A=2r$ and $B=-r^2$, $Ar+2B=0$.  Thus the last quantity in the chain of inequalities above is

$$kr^{k-2}(Ar+B) - r^{k-2}(Ar+2B) = \mathbf{kr^k}$$

This shows that the sequence *{nr$^n$} is a solution to the recurrence.*
(*{r$^n$} is also a solution*, which can be shown in the same way as before.)

* Using the foregoing facts, it can be shown, with a routine proof, that when the characteristic equation has a double root, *all solutions* to the recurrence *are linear combinations of {nr$^n$} and {r$^n$}*, and a solution exists for every pair of initial values $z_0$ and $z_1$.

**Sample illustrative problems for section 5.8**

* Ex 5.8.1: Which of the examples are second-order linear homogeneous recurrence relations with constant coefficients?

* Ex 5.8.8: (a) find the sequences, based on roots of the characteristic equation, that are solutions to the recurrence relation; and (b) find an explicit formula that satisfies both the initial conditions and the recurrence relation.

* Ex 5.8.11: Find an explicit formula for the given recurrence relation and initial conditions.

* Ex 5.8.13: Find an explicit formula for the given recurrence relation and initial conditions.

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**Day Twenty-Six: Wed, Nov 25, 2015**
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  * **Take Roll**
  * **Handback HW #19**

**Topics in Section 5.9 (General Recursive Definitions and Structural Induction)**

* Recursive Definition of a Set:

I. BASE: A statement that certain objects belong to the set
   Example: "Each symbol of the alphabet is a Boolean expression"

II. RECURSION: A collection of rules indicating how to form new set objects
  from those already known to be in the set.
   Example: "If P and Q are Boolean expressions, then so are
                (a) (P ∧ Q) and (b) (P ∨ Q) and (c) ~P."

III. RESTRICTION: A statement that no objects belong to the set other than
  those coming from I and II.
   Example: "There are no Boolean expressions over the alphabet other than
  those obtained from I and II."

* Definition of a ***string:*** Let S be a non-empty finite set.  ***A string over S***
  is a finite sequence of elements of S.  The elements are called ***characters***
  of the string, and the number of characters in the string is the ***length of***
  ***the string.***  The **null string** (aka ***empty string***) is the string with no
  characters.  The null string has **length 0** and is often denoted as **ε**
  (epsilon).

* ***The Structural Induction form of mathematical induction:***
  Let S be a set that has been defined recursively, and consider a
  proposition (statement) P(x) that may be true or false about objects x in
  S.  To prove that P(x) is true for every x in S:
     1. Show that P(x) is true for every object x in the BASE of S.
     2. Show that for each rule in the RECURSION, if the rule is applied
        to an object x in S for which P(x) is true, then P(y) is true for
        the object y defined by the rule.
  Because no objects other than those obtained through the BASE and
  RECURSION conditions are contained in S, it must be the case that P(x) is
  true for every object x in S.

**Sample illustrative problem for section 5.9**

* 5.9.7: Define S recursively by
I. BASE: ε ∈ S
II. RECURSION: if s ∈ S, then
      bs ∈ S, sb ∈ S, saa ∈ S, and aas ∈ S
III. RESTRICTION: Nothing is in S other than objects defined in I and II
   above.
Use structural induction to prove that every string in S contains an even
   number of a's.

**Topics in Section 6.1 (Set Theory: Definitions and the Element Method of
   Proof)**

**\* The Element Argument: to prove that X ⊆ Y, where X and Y are sets:**
```
     1. Suppose x is a particular but arbitrarily chosen element of X, and
     2. Show that x is an element of Y
```

**\* Set Equality:**
```
  Given sets A and B, A=B if and only if (A ⊆ B and B ⊆ A)
```

**\* Set Operations**
```
  Suppose A and B are subsets of some 'universal set' U
   The union A ∪ B of A and B is the set of all elements of U that are in
   either A or B, or both.
   The intersection A ∩ B of A and B is the set of all elements that are in
   both A and B.
   The difference A-B is the set of all elements of A that are not elements
   of B.
   The complement Aᶜ of A in U is the difference U-A.
```

**\* Interval Notation for real numbers:**
```
   Given a,b in R with a≤b
  (a,b) = {x∈R|a<x<b}
  [a,b] = {x∈R|a≤x≤b}
  [a,b) = {x∈R|a≤x<b}
  (a,b] = {x∈R|a<x≤b}
  (a, ∞) = {x∈R|a<x}
  [a, ∞) = {x∈R|a≤x}
  (-∞,b) = {x∈R|x<b}
  (-∞,b] = {x∈R|x≤b}
```

**\* Unions and Intersections of an Indexed Collection of Sets**

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**• Definition**

**Unions and Intersections of an Indexed Collection of Sets**

Given sets $A_0$, $A_1$, $A_2$, ... that are subsets of a universal set $U$ and given a nonnegative integer $n$,

$$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, \ldots, n\}$$

$$\bigcup_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for at least one nonnegative integer } i\}$$

$$\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for all } i = 0, 1, 2, \ldots, n\}$$

$$\bigcap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for all nonnegative integers } i\}.$$

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* **Alternative Notations: e.g. $A_0 \cup A_1 \cup \ldots \cup A_n$**

* **The Empty Set.**   The empty set is the set with no elements.   There is only
    one empty set.   It is denoted $\varnothing$.

* **Disjoint Sets.**   Two sets are disjoint if their intersection is the empty
    set.   Example: (1,2) and (3,4) are disjoint.

* **Pairwise Disjoint Collections of Sets** (aka **Mutually Disjoint**, aka **Non-
    overlapping**): A collection of sets $A_0$, $A_1$, $A_2$, ... $A_n$ is mutually disjoint
    if $A_i \cap A_j = \varnothing$ when ever $i \neq j$.

* **A Partition of Sets:** A finite or infinite collection of non-empty sets $A_0$,
    $A_1$, $A_2$, ... is a partition of a set A if and only if
        1. A is the union of all the $A_i$.
        2. The sets $A_0$, $A_1$, $A_2$, ... are pairwise disjoint.

* **Power Sets:** The power set $\mathcal{P}$(A) of the set A is the set of all subsets of A.

* **Cartesian Products**

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• **Definition**

Let $n$ be a positive integer and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct)
elements. The **ordered $n$-tuple, $(x_1, x_2, \ldots, x_n)$,** consists of $x_1, x_2, \ldots, x_n$ together
with the ordering: first $x_1$, then $x_2$, and so forth up to $x_n$. An ordered 2-tuple is called
an **ordered pair,** and an ordered 3-tuple is called an **ordered triple.**

Two ordered $n$-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are **equal** if, and only
if, $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$.

Symbolically:

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \iff x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n.$$

In particular,

$$(a, b) = (c, d) \iff a = c \text{ and } b = d.$$

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• **Definition**

Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$ denoted
$A_1 \times A_2 \times \ldots \times A_n$, is the set of all ordered $n$-tuples $(a_1, a_2, \ldots, a_n)$ where
$a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

In particular,

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$$

is the Cartesian product of $A_1$ and $A_2$.

**Math 2300, Section #1; 14:00-15:15 MW P-101; Daily Notes Fall, 2015**


**Sample illustrative problem for section 6.1 (Definitions and the Element Method of Proof)**

  * Take Roll
  * Hand back Quiz #2 - go over answers


* Ex 6.1.4: A = integral multiples of 5, B = integral multiples of 20.
      Is $A \subseteq B$?  Is $B \subseteq A$?

* Ex 6.1.11: Intervals A=(0,2]; B=[1,4); C=[3,9)
      Find various unions, intersections, and complements involving A,B,C.
(a) $(A \cup B)$ = ? ; (b) $(A \cap B)$ = ?; (c) $(A^C)$ = ? (d) $(A \cup C)$ = ?;
(e) $(A \cap C)$ = ?; (f) $(B^C)$ = ?; (g) $(A^C \cap B^C)$ = ?; (h) $(A^C \cup B^C)$ = ?;
(i) $((A \cap B)^C)$ = ?; (j) $((A \cup B)^C)$ = ?;

* Ex 6.1.15
   Draw Venn Diagrams to describe sets satisfying certain conditions
(a) $(A \cap B) = \varnothing$, $A \subseteq C$, $(C \cap B) \neq \varnothing$
(b) $A \subseteq B$, $C \subseteq B$, $(C \cap B) \neq \varnothing$
(c) $(A \cap B) \neq \varnothing$, $(B \cap C) \neq \varnothing$, $(A \cap C) = \varnothing$, $A \not\subseteq B$, $C \not\subseteq B$

* 6.1.27 (a) & (d)
(a) Is {{a,d,e}, {b,c}, {d,f}} a partition of {a,b,c,d,e,f}?
(d) Is {{3,7,8}, {2,9}, {1,4,5}} a partition of {1,2,3,4,5,6,7,8,9}?

* 6.1.31: Find various power sets
A = {1,2} ; B = {2,3}
Find:

$\mathcal{P}(A \cap B), \; \mathcal{P}(A), \; \mathcal{P}(A \cup B), \; \mathcal{P}(A \times B),$

**Topics in Section 6.2 (Properties of Sets)**

**Theorem 6.2.1 Some Subset Relations**

1. *Inclusion of Intersection:* For all sets $A$ and $B$,

$$(a)\ A \cap B \subseteq A \quad \text{and} \quad (b)\ A \cap B \subseteq B.$$

2. *Inclusion in Union:* For all sets $A$ and $B$,

$$(a)\ A \subseteq A \cup B \quad \text{and} \quad (b)\ B \subseteq A \cup B.$$

3. *Transitive Property of Subsets:* For all sets $A$, $B$, and $C$,

$$\text{if } A \subseteq B \text{ and } B \subseteq C, \text{ then } A \subseteq C.$$

**Procedural Versions of Set Definitions**

Let $X$ and $Y$ be subsets of a universal set $U$ and suppose $x$ and $y$ are elements of $U$.

1. $x \in X \cup Y \quad \Leftrightarrow \quad x \in X$ or $x \in Y$

2. $x \in X \cap Y \quad \Leftrightarrow \quad x \in X$ and $x \in Y$

3. $x \in X - Y \quad \Leftrightarrow \quad x \in X$ and $x \notin Y$

4. $x \in X^c \quad \Leftrightarrow \quad x \notin X$

5. $(x, y) \in X \times Y \quad \Leftrightarrow \quad x \in X$ and $y \in Y$

**Theorem 6.2.2 Set Identities**

Let all sets referred to below be subsets of a universal set $U$.

1. *Commutative Laws:* For all sets $A$ and $B$,

$$\text{(a) } A \cup B = B \cup A \quad \text{and} \quad \text{(b) } A \cap B = B \cap A.$$

2. *Associative Laws:* For all sets $A$, $B$, and $C$,

$$\text{(a) } (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$
$$\text{(b) } (A \cap B) \cap C = A \cap (B \cap C).$$

3. *Distributive Laws:* For all sets, $A$, $B$, and $C$,

$$\text{(a) } A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$
$$\text{(b) } A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

4. *Identity Laws:* For all sets $A$,

$$\text{(a) } A \cup \emptyset = A \quad \text{and} \quad \text{(b) } A \cap U = A.$$

5. *Complement Laws:*

$$\text{(a) } A \cup A^c = U \quad \text{and} \quad \text{(b) } A \cap A^c = \emptyset.$$

6. *Double Complement Law:* For all sets $A$,

$$(A^c)^c = A.$$

7. *Idempotent Laws:* For all sets $A$,

$$\text{(a) } A \cup A = A \quad \text{and} \quad \text{(b) } A \cap A = A.$$

8. *Universal Bound Laws:* For all sets $A$,

$$\text{(a) } A \cup U = U \quad \text{and} \quad \text{(b) } A \cap \emptyset = \emptyset.$$

9. *De Morgan's Laws:* For all sets $A$ and $B$,

$$\text{(a) } (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b) } (A \cap B)^c = A^c \cup B^c.$$

10. *Absorption Laws:* For all sets $A$ and $B$,

$$\text{(a) } A \cup (A \cap B) = A \quad \text{and} \quad \text{(b) } A \cap (A \cup B) = A.$$

11. *Complements of $U$ and $\emptyset$:*

$$\text{(a) } U^c = \emptyset \quad \text{and} \quad \text{(b) } \emptyset^c = U.$$

12. *Set Difference Law:* For all sets $A$ and $B$,

$$A - B = A \cap B^c.$$

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*   How to prove two sets A, B are equal: Prove (A ⊆ B) and (B ⊆ A)
*   Theorem 6.2.4: A set with no elements is a subset of very set.
*   There is only one set with no elements, <u>the</u> empty set.
*   How to prove a set is empty by contradiction: Assume it has an element and derive a contradiction.

**Sample illustrative problem for section 6.2 (Properties of Sets)**

* 6.2.3: Fill in the blanks of the following proof that ∀ sets A, B, C, when (A ⊆ B) and (B ⊆ C), (A ⊆ C).
Proof: Suppose A, B, and C are sets, and that (A ⊆ B) and (B ⊆ C).  To show that (A ⊆ C), we must show that every element in ____(a)____ is in ___(b)____.  But given any element in A, that element is in ____(c)_____ (because A ⊆ B), and so that element is also in ____(d)____ (because ____(e)____).  Hence (A ⊆ C).

* 6.2.10: Use an element argument to prove the statement.  Assume all sets are subsets of a universal set U.
For all sets A, B, and C, (A – B) ∩ (C – B) = (A ∩ C) – B

* 6.2.16: Use an element argument to prove the statement.  Assume all sets are subsets of a universal set U.
For all sets A, B, and C, if (A ⊆ B) and (A ⊆ C) then ( A ⊆ (B ∩ C) )

* 6.2.34: Use the element method for proving the set equals the empty set. (i.e. Suppose the set has an element and derive a contradiction.) Assume that all sets are subsets of a universal set U.
For all sets A, B, and C, if (B ∩ C) ⊆ A, then (C – A) ∩ (B – A) = ∅

**Topics in Section 6.3 (Disproofs, Algebraic Proofs, and Boolean Algebras)**
**\* A way to construct a counter example to a false set identity**
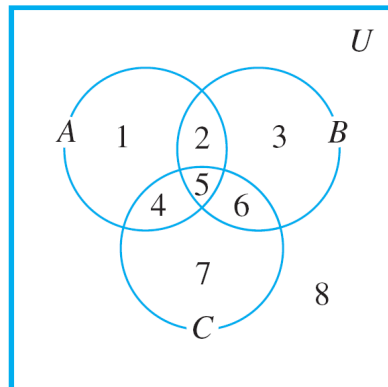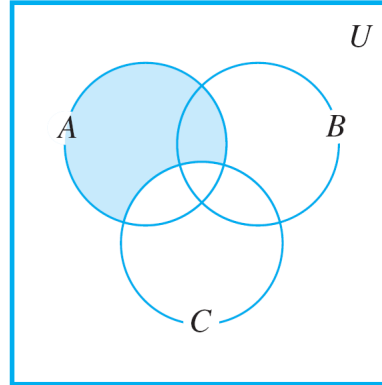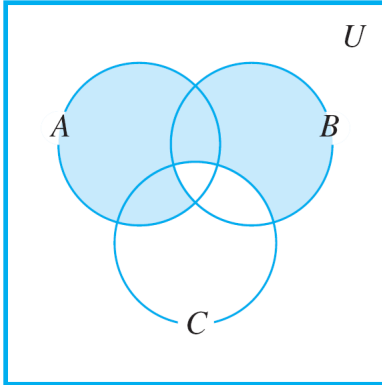


We check the diagram to ascertain the plausibility that

(A - B) ∪ (B - C) = A - C

is a correct set identity.  Using the labels of the sections to construct
   example sets, we get A = {1,2,4,5}, B = {2,3,5,6}, and C={4,5,6,7}.  Here
   A - C = {1,2}, A - B = {1,4}, and B - C = {2,3}

**\* Proof by induction on 0≤n∈Z that a set with n elements has $2^n$ subsets.**

Sample illustrative problem for section 6.3 (Disproofs, Algebraic Proofs, and
   Boolean Algebras)

**\* 6.3.1: Find a counter-example: ∀ sets A, B, and C, (all contained in a
   universal set, U)         (A ∩ B) ∪ C = A ∩ (B ∪ C)**

* 6.3.17: Prove or give a counter-example: ∀ sets A, B (contained in a

   universal set U): if A ⊂ B then $\mathcal{P}(A) \subset \mathcal{P}(B)$.

* 6.3.18: Prove or give a counter-example: ∀ sets A, B (contained in a
   universal set U):

* 6.3.34: Construct an algebraic proof (a proof that uses only the identities
   of theorem 6.2.2.)
       Prove: ∀ sets A, B, C (contained in a universal set U):
          (A-B)-C = A-(B ∪ C)

**Topics in Section 8.1 (Relations on Sets)**

* Examples of binary relations: x≤y in RxR, and "m-n is even" in ZxZ.
* The inverse of a binary relation: If R is a relation on AxB, then $R^{-1}$ is
   the relation on BxA given by (b,a) ∈ $R^{-1}$ if and only if (a,b)∈ R.
* A relation on a set A is a subset of AxA.  It can be represented as a
   directed graph with a set of nodes representing the elements of A.
* An N-ary relation - a relation on the cross product of N sets - is a
   subset of that cross product.  As objects of logical and mathematical
   study, relational databases are viewed as N-ary relations.


**Sample illustrative problem for section 8.1 (Relations on Sets)**

* 8.1.1: The congruence relation E from Z to Z is defined by
   m E n <--> (m-n) is even, ∀ m,n ∈ Z.
       a) Which of these are true? 0E0, 5E2, (6,6)∈E, (-1,7)∈E
       b) Prove that nE0 is true for any even integer n.

* 8.1.9: Let A be the set of all strings of length 4 made of 0's 1's and
   2's.  Define the relation R on A by: sRt <--> the sum of the characters in
   s is equal to the sum of the characters in t.
       a) is 0121 R 2200?  What about 1011 R 2101, 2212 R 2121, and 1220 R
   2111?

* 8.1.10: Let A = {3,4,5} and B = {4,5,6} and let R be the "less than"
   relation on AxB.  State explicitly which ordered pairs are in R and $R^{-1}$.

**Topics in Section 8.2 (Reflexivity, Symmetry, and Transitivity)**

*   Definition of reflexive relation on a set, symmetric relation on a set,
    and transitive relation on a set
*   The relation of "equality" on the set of real numbers is reflexive,
    symmetric, and transitive.  (Any 'reasonably defined' equality relation
    will be reflexive, symmetric, and transitive.  For example equality of
    sets is reflexive, symmetric, and transitive.)

* The "less than" relation on real numbers is transitive.  The "less than or
    equal" relation on real numbers and the set containment relation are
    reflexive and transitive, but not symmetric.

*   Congruence of integers modulo n is reflexive, symmetric, and transitive.

*   Given a relation that is reflexive and symmetric, the relation can be
    extended to its "transitive closure" -- by adding all the necessary pairs
    to the relation to make it transitive.