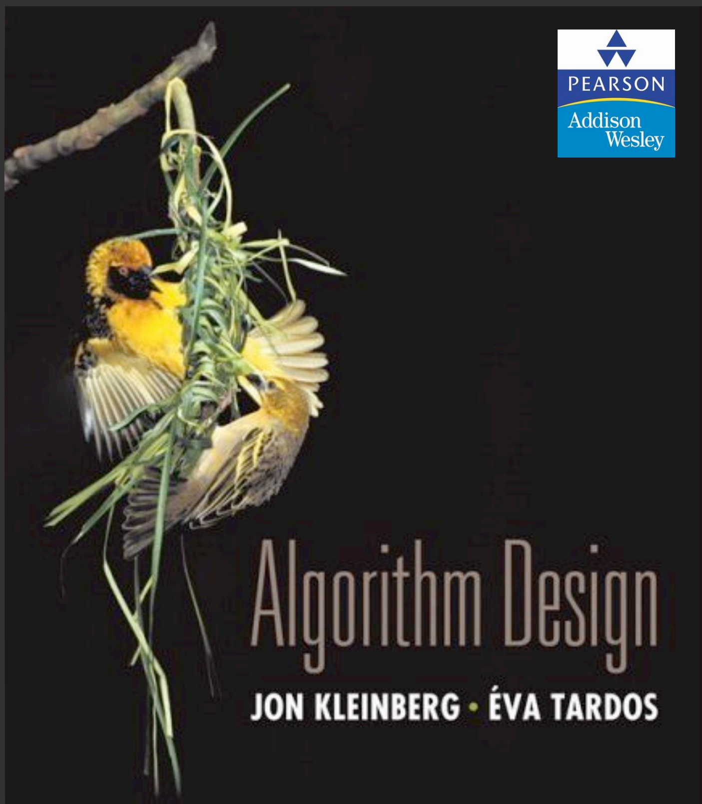


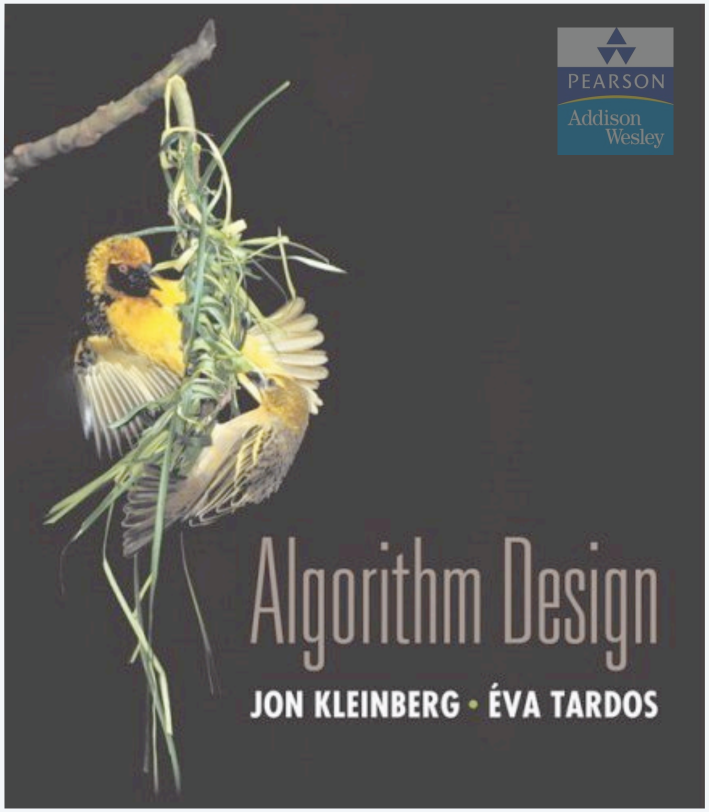
6. DYNAMIC PROGRAMMING II

- *sequence alignment*
- *Hirschberg's algorithm*
- *Bellman-Ford algorithm*
- *distance vector protocols*
- *negative cycles in a digraph*

Lecture slides by Kevin Wayne

http://www.cs.princeton.edu/~wayne/kleinberg-tardos

Last updated on Sep 8, 2013 6:33 AM



SECTION 6.6

6. DYNAMIC PROGRAMMING II

- *sequence alignment*
- *Hirschberg's algorithm*
- *Bellman-Ford algorithm*
- *distance vector protocols*
- *negative cycles in a digraph*

String similarity

Q. How similar are two strings?

Ex. ocurrance and occurrence.

o	c	u	r	r	a	n	c	e	–
o	c	c	u	r	r	e	n	c	e

6 mismatches, 1 gap

o	c	–	u	r	r	a	n	c	e
o	c	c	u	r	r	e	n	c	e

1 mismatch, 1 gap

o	c	–	u	r	r	–	a	n	c	e
o	c	c	u	r	r	e	–	n	c	e

0 mismatches, 3 gaps

Edit distance

Edit distance. [Levenshtein 1966, Needleman-Wunsch 1970]

- Gap penalty δ; mismatch penalty α_{pq}.
- Cost = sum of gap and mismatch penalties.

C	T	–	G	A	C	C	T	A	C	G
C	T	G	G	A	C	G	A	A	C	G

cost $= \boldsymbol{\delta} + \boldsymbol{\alpha}_{\mathrm{CG}} + \boldsymbol{\alpha}_{\mathrm{TA}}$

Applications. Unix diff, speech recognition, computational biology, ...

Sequence alignment

Goal. Given two strings $x_1\, x_2 \ldots x_m$ and $y_1\, y_2 \ldots y_n$ find min cost alignment.

Def. An alignment M is a set of ordered pairs $x_i - y_j$ such that each item occurs in at most one pair and no crossings.

$x_i - y_j$ and $x_{i'} - y_{j'}$ cross if $i < i'$, but $j > j'$

Def. The cost of an alignment M is:

$$\text{cost}(M) = \underbrace{\sum_{(x_i, y_j) \in M} \alpha_{x_i y_j}}_{\text{mismatch}} + \underbrace{\sum_{i\,:\,x_i \text{ unmatched}} \delta + \sum_{j\,:\,y_j \text{ unmatched}} \delta}_{\text{gap}}$$

x_1	x_2	x_3	x_4	x_5		x_6
C	T	A	C	C	–	G
–	T	A	C	A	T	G
	y_1	y_2	y_3	y_4	y_5	y_6

an alignment of CTACCG and TACATG:

$$M = \{\, x_2\text{–}y_1,\, x_3\text{–}y_2,\, x_4\text{–}y_3,\, x_5\text{–}y_4,\, x_6\text{–}y_6 \,\}$$

Sequence alignment: problem structure

Def. $OPT(i, j)$ = min cost of aligning prefix strings $x_1\, x_2 \ldots x_i$ and $y_1\, y_2 \ldots y_j$.

Case 1. OPT matches $x_i - y_j$.
Pay mismatch for $x_i - y_j$ + min cost of aligning $x_1\, x_2 \ldots x_{i-1}$ and $y_1\, y_2 \ldots y_{j-1}$.

Case 2a. OPT leaves x_i unmatched.
Pay gap for x_i + min cost of aligning $x_1\, x_2 \ldots x_{i-1}$ and $y_1\, y_2 \ldots y_j$.

optimal substructure property
(proof via exchange argument)

Case 2b. OPT leaves y_j unmatched.
Pay gap for y_j + min cost of aligning $x_1\, x_2 \ldots x_i$ and $y_1\, y_2 \ldots y_{j-1}$.

$$OPT(i, j) = \begin{cases} j\delta & \text{if } i = 0 \\ \min \begin{cases} \alpha_{x_i y_j} + OPT(i-1, j-1) \\ \delta + OPT(i-1, j) \\ \delta + OPT(i, j-1) \end{cases} & \text{otherwise} \\ i\delta & \text{if } j = 0 \end{cases}$$

Sequence alignment: algorithm

SEQUENCE-ALIGNMENT $(m, n, x_1, \ldots, x_m, y_1, \ldots, y_n, \delta, \alpha)$

FOR $i = 0$ TO m

  $M[i, 0] \leftarrow i\,\delta$.

FOR $j = 0$ TO n

  $M[0, j] \leftarrow j\,\delta$.

FOR i = 1 TO m

  FOR j = 1 TO n

$$M[i, j] \leftarrow \min \{\ \alpha[x_i, y_j] + M[i-1, j-1],\ \delta + M[i-1, j],\ \delta + M[i, j-1]).$$

RETURN $M[m, n]$.

Sequence alignment: analysis

Theorem. The dynamic programming algorithm computes the edit distance (and optimal alignment) of two strings of length m and n in $\Theta(mn)$ time and $\Theta(mn)$ space.

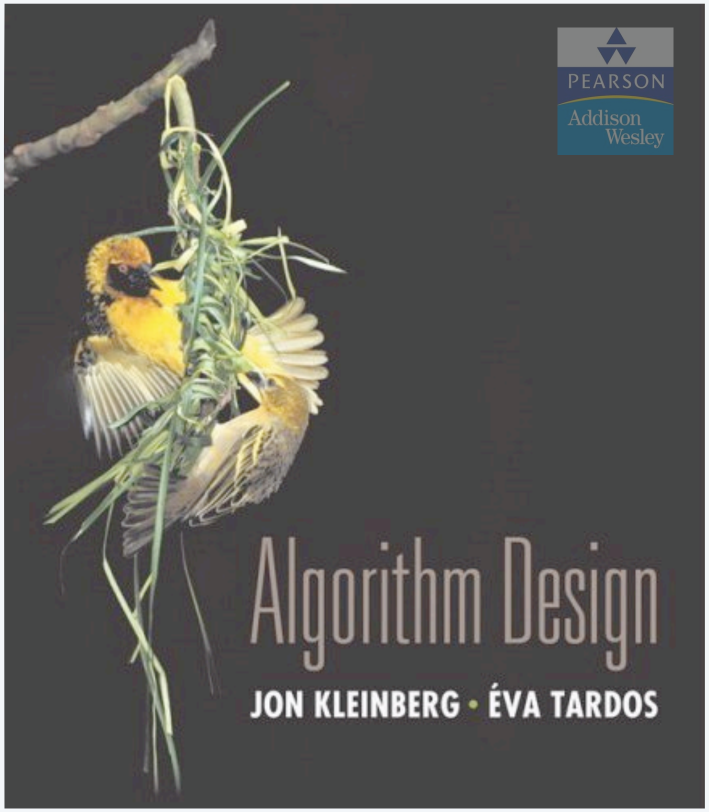
Pf.

- Algorithm computes edit distance.
- Can trace back to extract optimal alignment itself. ▪

Q. Can we avoid using quadratic space?

A. Easy to compute optimal value in $O(mn)$ time and $O(m + n)$ space.

- Compute $\text{OPT}(i, \bullet)$ from $\text{OPT}(i - 1, \bullet)$.
- But, no longer easy to recover optimal alignment itself.



SECTION 6.7

6. DYNAMIC PROGRAMMING II

- *sequence alignment*
- *Hirschberg's algorithm*
- *Bellman-Ford algorithm*
- *distance vector protocols*
- *negative cycles in a digraph*

Sequence alignment in linear space

Theorem. There exist an algorithm to find an optimal alignment in $O(mn)$ time and $O(m+n)$ space.

- Clever combination of divide-and-conquer and dynamic programming.
- Inspired by idea of Savitch from complexity theory.

Programming Techniques — G. Manacher, Editor

A Linear Space Algorithm for Computing Maximal Common Subsequences

D.S. Hirschberg
Princeton University

The problem of finding a longest common subsequence of two strings has been solved in quadratic time and space. An algorithm is presented which will solve this problem in quadratic time and in linear space.

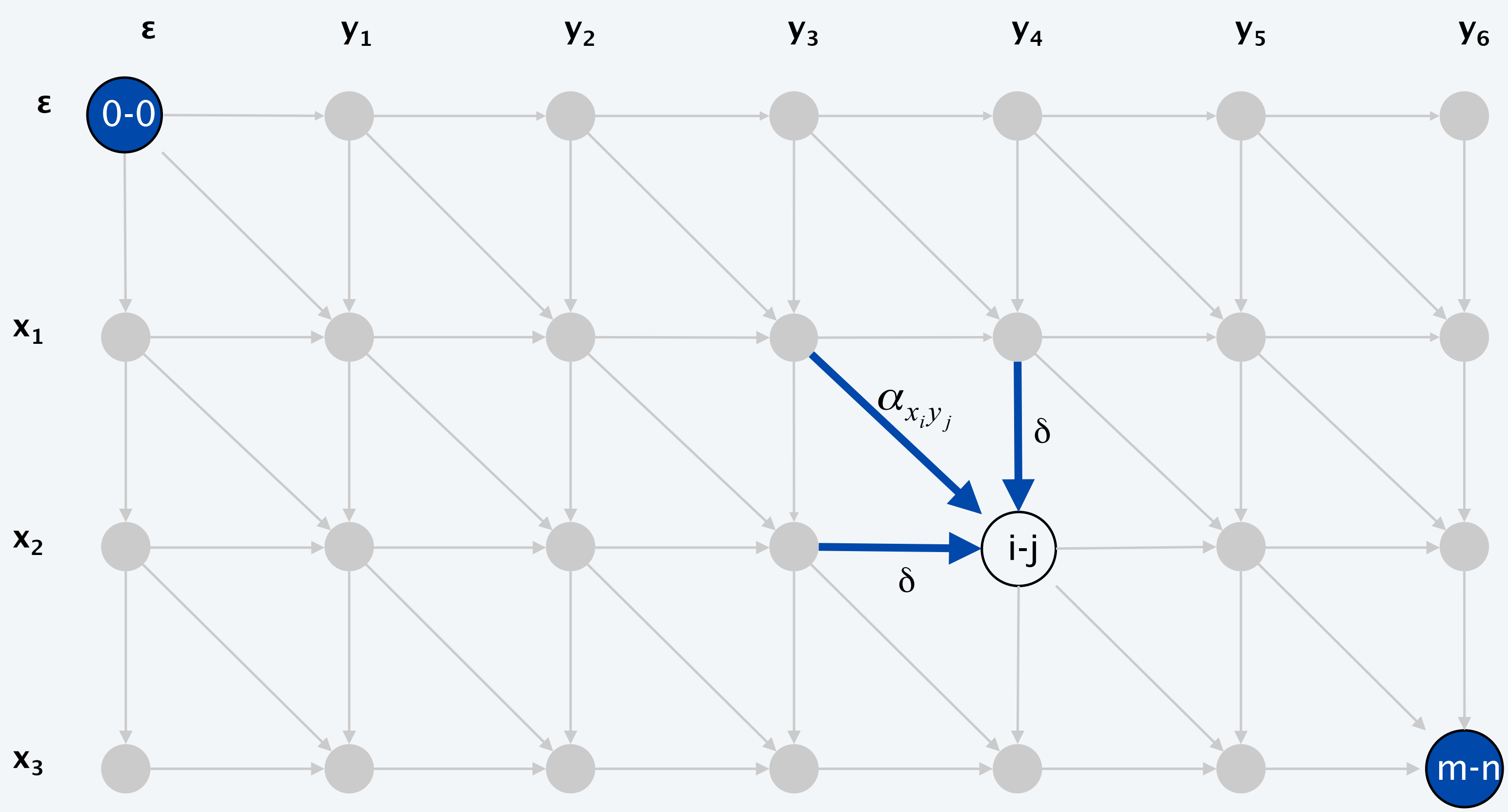
Key Words and Phrases: subsequence, longest common subsequence, string correction, editing

CR Categories: 3.63, 3.73, 3.79, 4.22, 5.25

Hirschberg's algorithm

Edit distance graph.

- Let $f(i, j)$ be shortest path from $(0,0)$ to (i, j).
- Lemma: $f(i, j) = OPT(i, j)$ for all i and j.



Hirschberg's algorithm

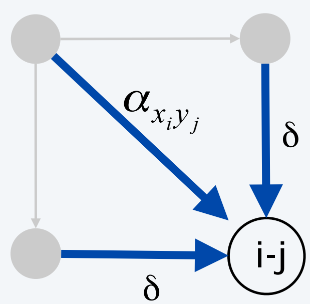
Edit distance graph.

- Let $f(i, j)$ be shortest path from $(0,0)$ to (i, j).
- Lemma: $f(i, j) = OPT(i, j)$ for all i and j.

Pf of Lemma. [by strong induction on $i + j$]

- Base case: $f(0, 0) = OPT(0, 0) = 0$.
- Inductive hypothesis: assume true for all (i', j') with $i' + j' < i + j$.
- Last edge on shortest path to (i, j) is from $(i - 1, j - 1)$, $(i - 1, j)$, or $(i, j - 1)$.
- Thus,

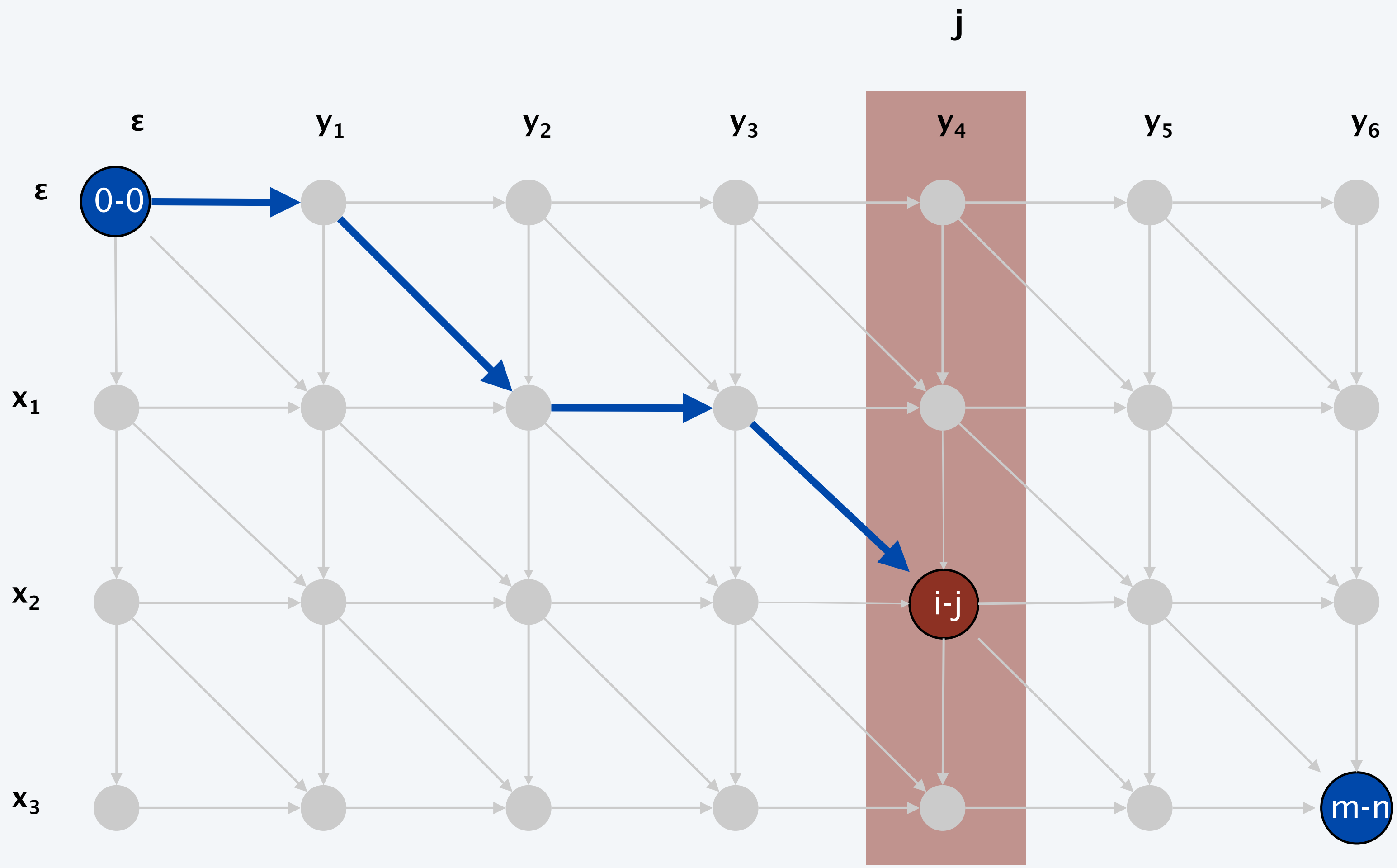
$$\begin{aligned} f(i,j) &= \min\{\alpha_{x_i y_j} + f(i-1, j-1),\ \delta + f(i-1, j),\ \delta + f(i, j-1)\} \\ &= \min\{\alpha_{x_i y_j} + OPT(i-1, j-1),\ \delta + OPT(i-1, j),\ \delta + OPT(i, j-1)\} \\ &= OPT(i,j) \quad \blacksquare \end{aligned}$$



Hirschberg's algorithm

Edit distance graph.

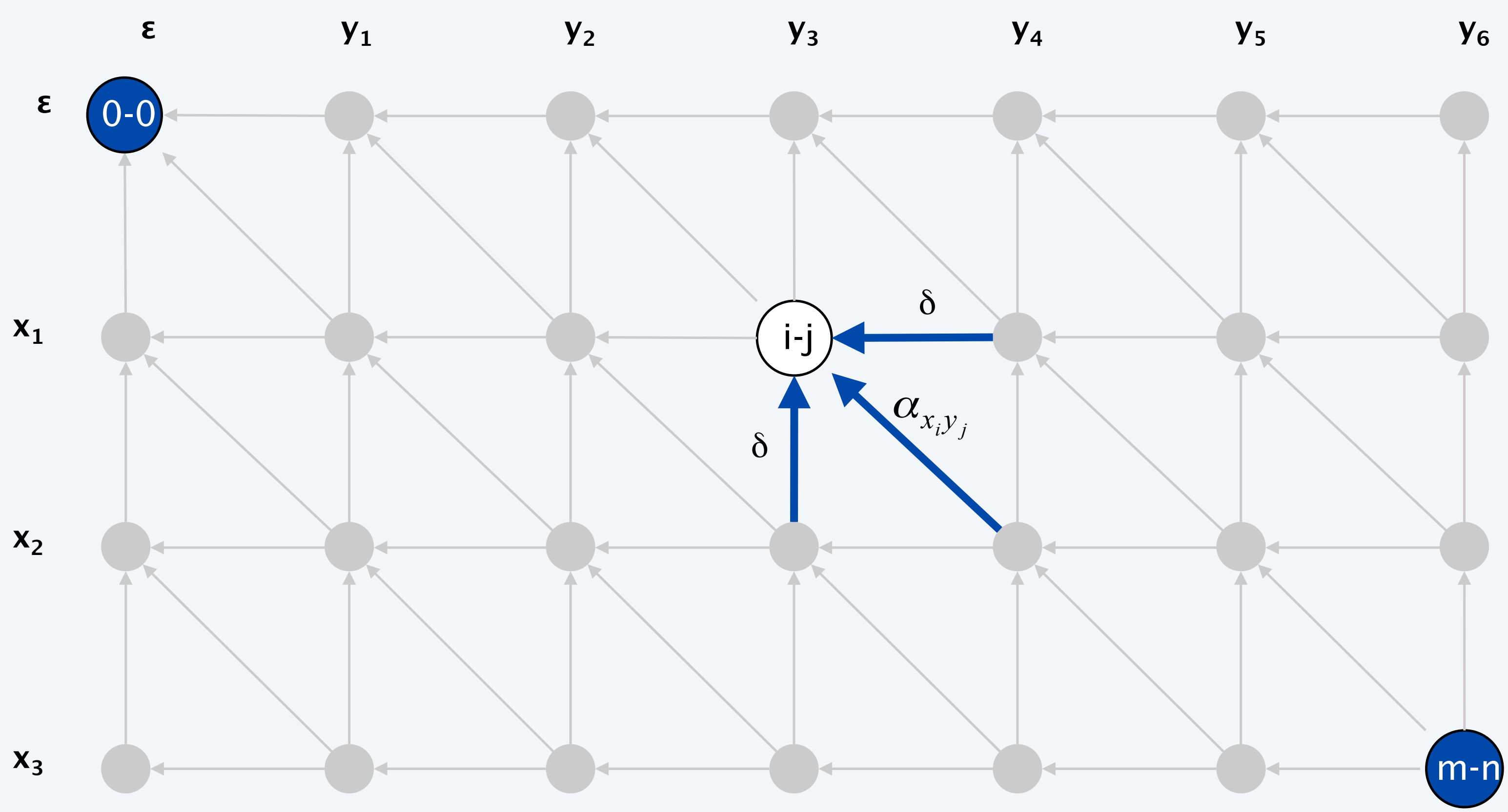
- Let $f(i, j)$ be shortest path from $(0,0)$ to (i, j).
- Lemma: $f(i, j) = OPT(i, j)$ for all i and j.
- Can compute $f(\bullet, j)$ for any j in $O(mn)$ time and $O(m + n)$ space.



Hirschberg's algorithm

Edit distance graph.

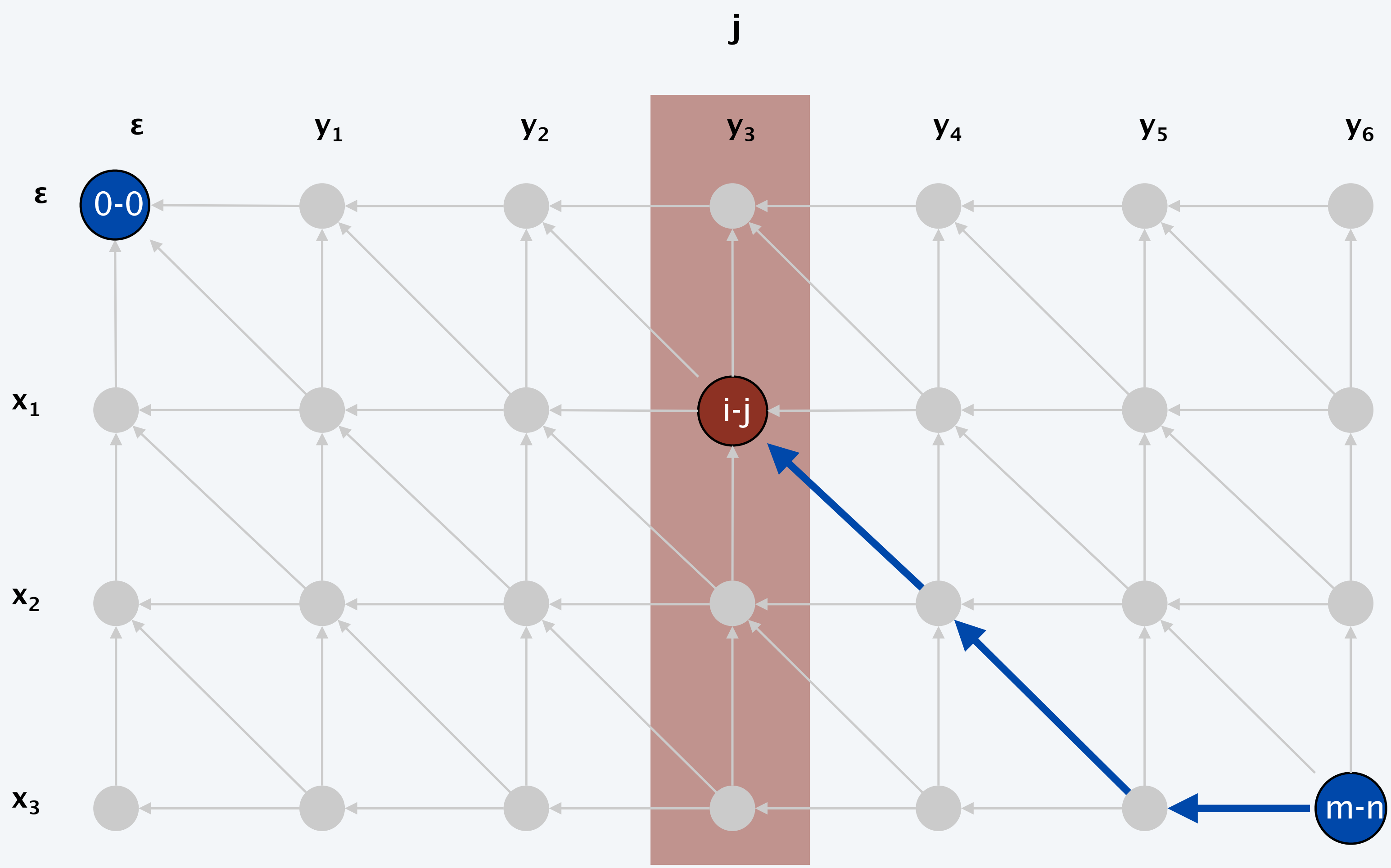
- Let $g(i, j)$ be shortest path from (i, j) to (m, n).
- Can compute by reversing the edge orientations and inverting the roles of $(0, 0)$ and (m, n).



Hirschberg's algorithm

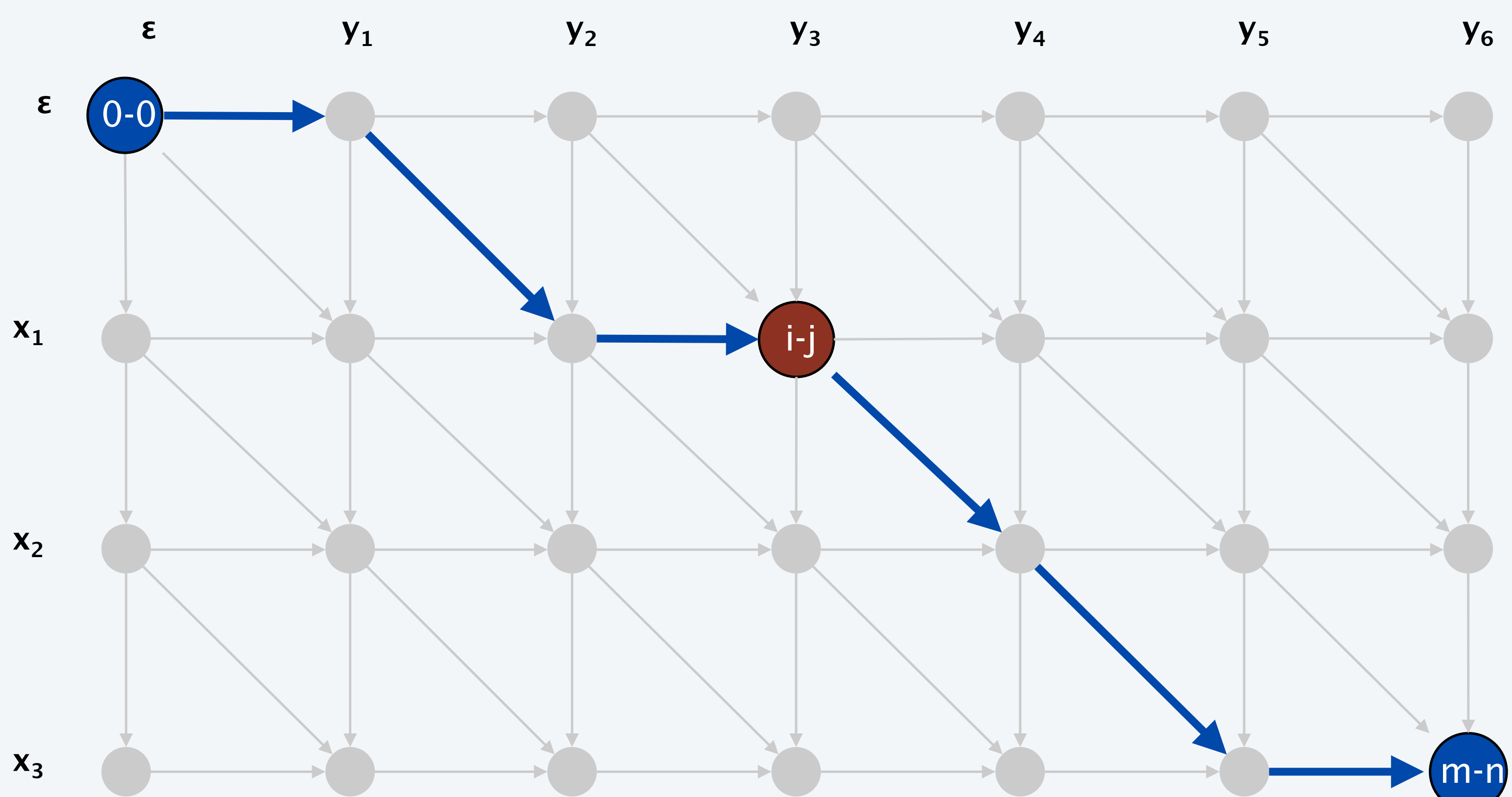
Edit distance graph.

- Let $g(i, j)$ be shortest path from (i, j) to (m, n).
- Can compute $g(\bullet,\ j)$ for any j in $O(mn)$ time and $O(m + n)$ space.



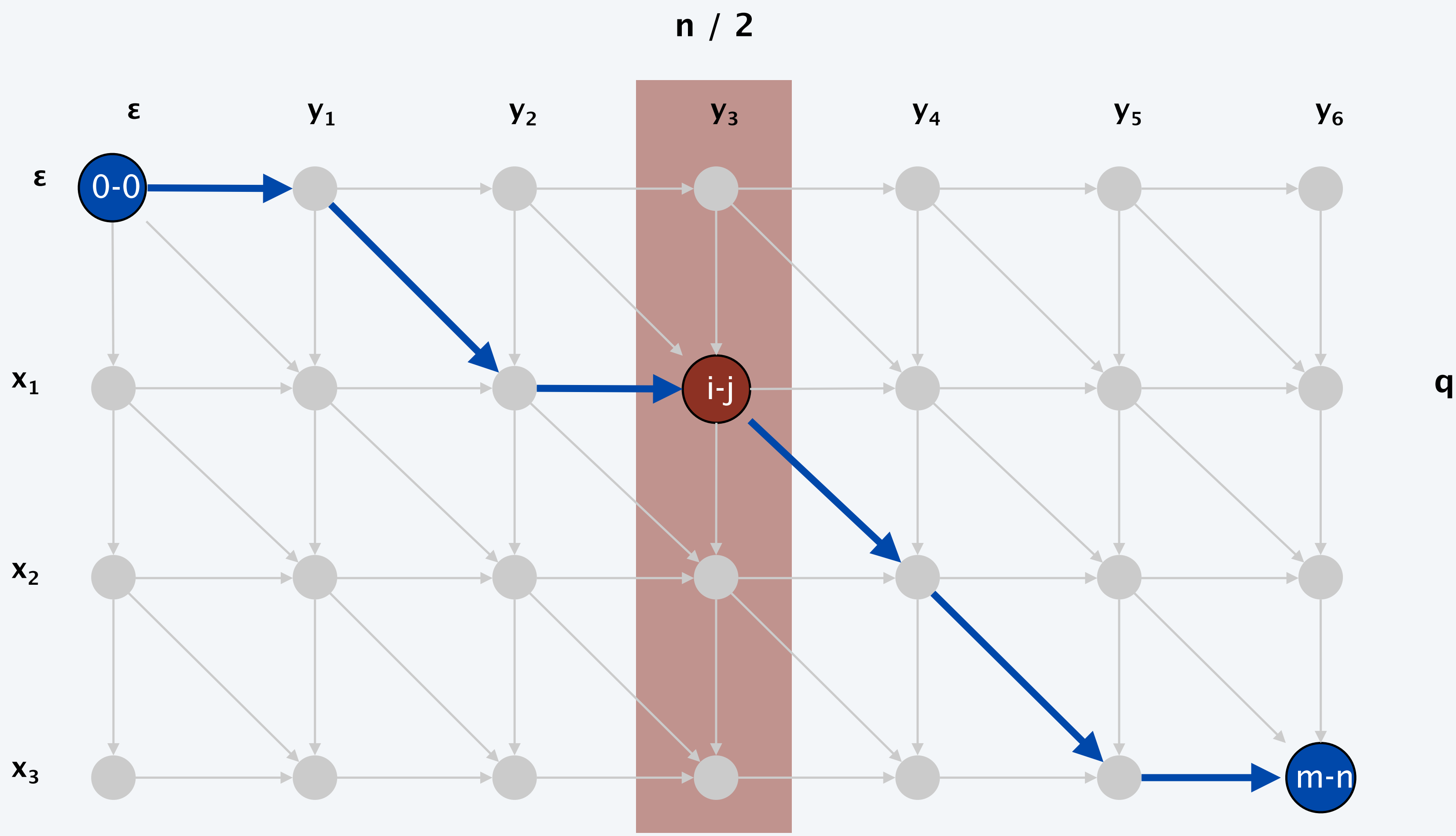
Hirschberg's algorithm

Observation 1. The cost of the shortest path that uses (i, j) is $f(i, j) + g(i, j)$.



Hirschberg's algorithm

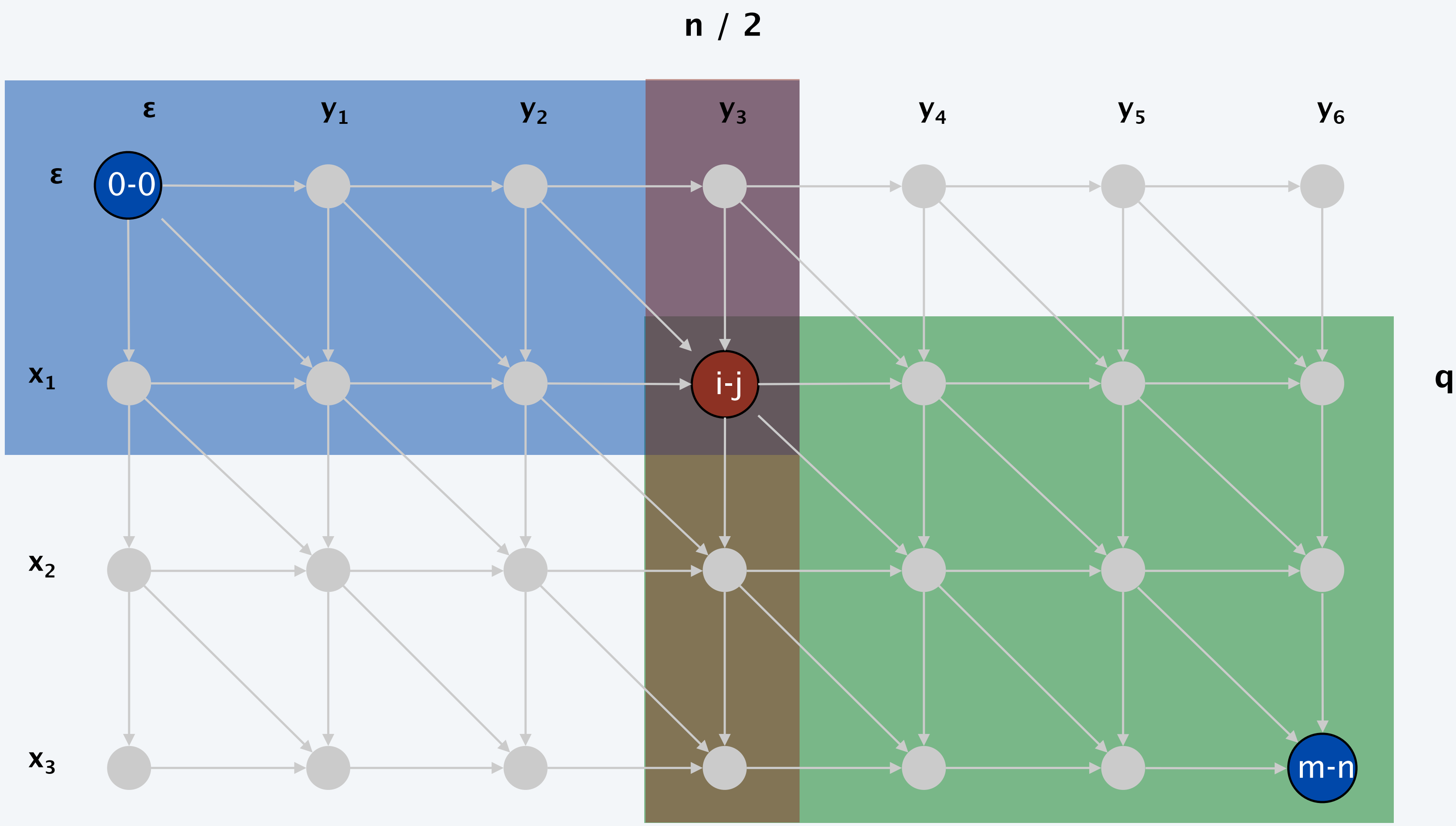
Observation 2. let q be an index that minimizes $f(q, n/2) + g(q, n/2)$. Then, there exists a shortest path from $(0, 0)$ to (m, n) uses $(q, n/2)$.



Hirschberg's algorithm

Divide. Find index q that minimizes $f(q, n/2) + g(q, n/2)$; align x_q and $y_{n/2}$.

Conquer. Recursively compute optimal alignment in each piece.



Hirschberg's algorithm: running time analysis warmup

Theorem. Let $T(m, n)$ = max running time of Hirschberg's algorithm on strings of length at most m and n. Then, $T(m, n) = O(m\,n \log n)$.

Pf. $T(m, n) \;\le\; 2\,T(m, n/2) \;+\; O(m\,n)$

$$\Rightarrow \quad T(m, n) = O(m \log n).$$

Remark. Analysis is not tight because two subproblems are of size $(q, n/2)$ and $(m - q, n/2)$. In next slide, we save $\log n$ factor.

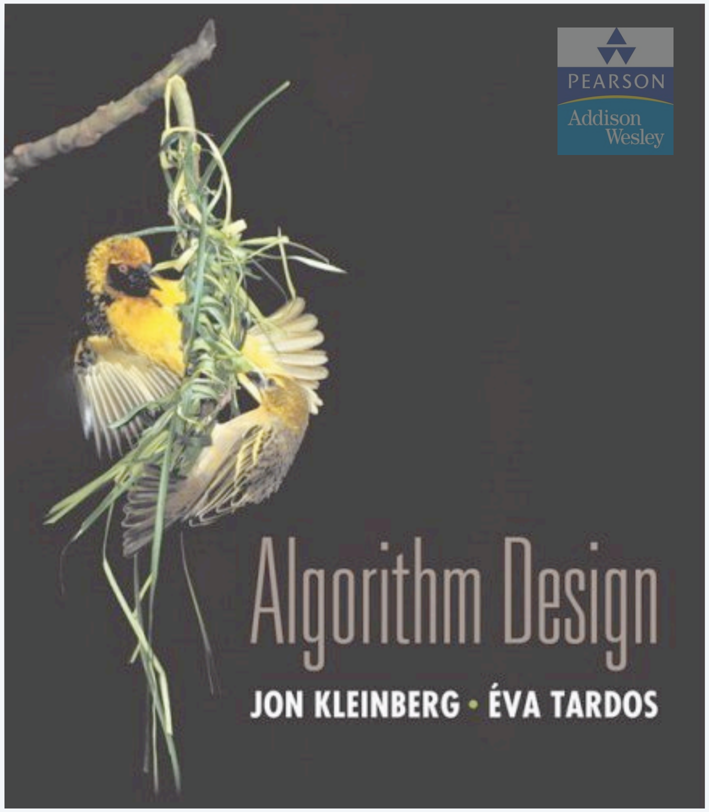
Hirschberg's algorithm: running time analysis

Theorem. Let $T(m, n)$ = max running time of Hirschberg's algorithm on strings of length at most m and n. Then, $T(m, n) = O(m n)$.

Pf. [by induction on n]

- $O(m n)$ time to compute $f(\bullet,\ n/2)$ and $g(\bullet,\ n/2)$ and find index q.
- $T(q, n/2) + T(m - q, n/2)$ time for two recursive calls.
- Choose constant c so that: $$\begin{aligned} T(m, 2) &\leq c\,m \\ T(2, n) &\leq c\,n \\ T(m, n) &\leq c\,m\,n + T(q, n/2) + T(m - q, n/2) \end{aligned}$$
- Claim. $T(m, n) \ \leq\ 2\,c\,m\,n$.
- Base cases: $m = 2$ or $n = 2$.
- Inductive hypothesis: $T(m, n) \ \leq\ 2\,c\,m\,n$ for all (m', n') with $m' + n' < m + n$.

$$\begin{aligned} T(m, n) &\leq T(q, n/2) + T(m - q, n/2) + c\,m\,n \\ &\leq 2\,c\,q\,n/2 \ +\ 2\,c\,(m - q)\,n/2 \ +\ c\,m\,n \\ &= c\,q\,n \ +\ c\,m\,n \ -\ c\,q\,n \ +\ c\,m\,n \\ &= 2\,c\,m\,n \ \blacksquare \end{aligned}$$



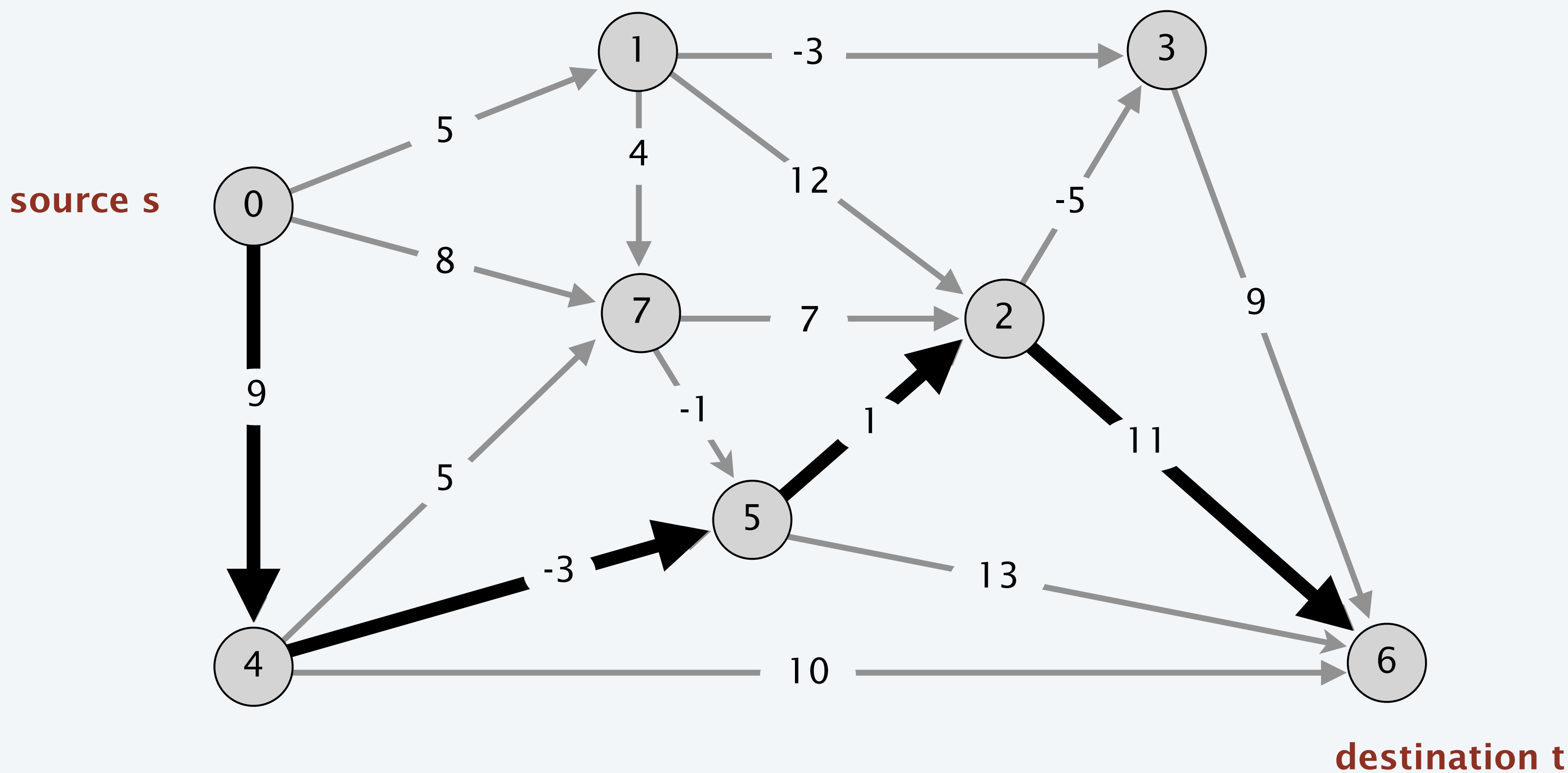
SECTION 6.8

6. DYNAMIC PROGRAMMING II

- *sequence alignment*
- *Hirschberg's algorithm*
- *Bellman-Ford*
- *distance vector protocols*
- *negative cycles in a digraph*

Shortest paths

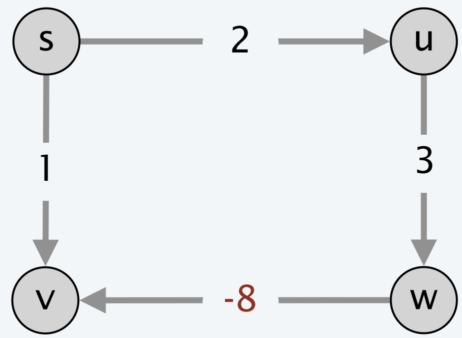
Shortest path problem. Given a digraph $G = (V, E)$, with arbitrary edge weights or costs c_{vw}, find cheapest path from node s to node t.



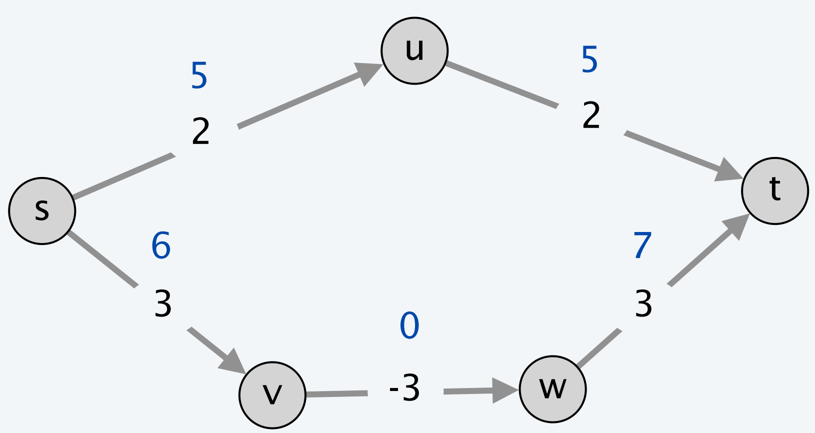
cost of path = 9 - 3 + 1 + 11 = 18

Shortest paths: failed attempts

Dijkstra. Can fail if negative edge weights.

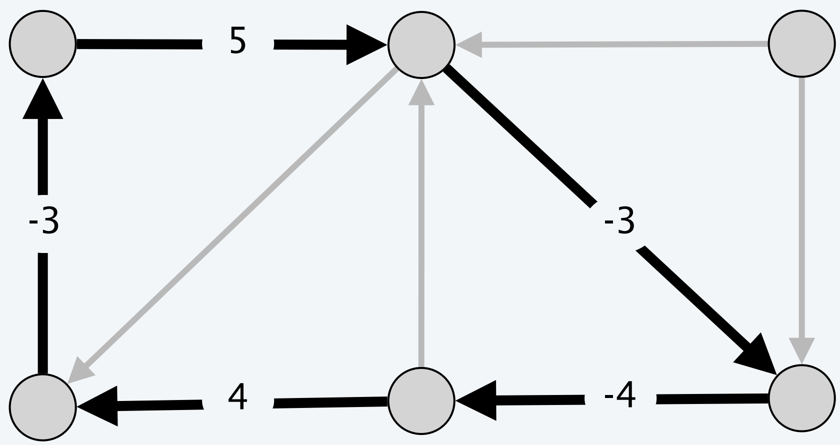


Reweighting. Adding a constant to every edge weight can fail.



Negative cycles

Def. A negative cycle is a directed cycle such that the sum of its edge weights is negative.

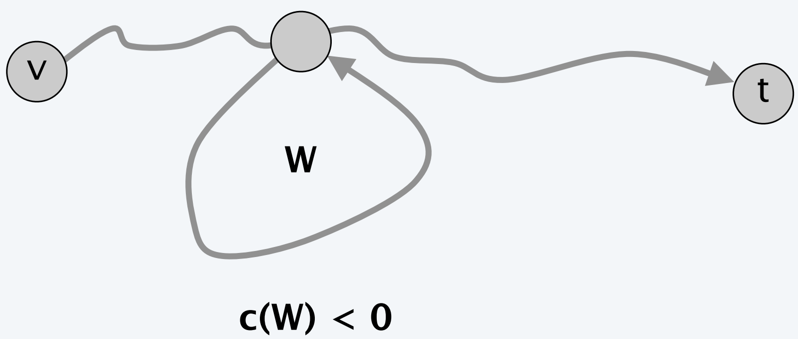


a negative cycle W : $c(W) \;=\; \sum_{e \in W} c_e \;<\; 0$

Shortest paths and negative cycles

Lemma 1. If some path from v to t contains a negative cycle, then there does not exist a cheapest path from v to t.

Pf. If there exists such a cycle W, then can build a $v \rightarrow t$ path of arbitrarily negative weight by detouring around cycle as many times as desired. ▪

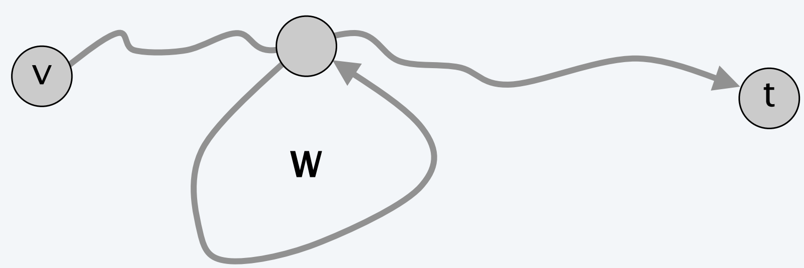


Shortest paths and negative cycles

Lemma 2. If G has no negative cycles, then there exists a cheapest path from v to t that is simple (and has $\leq n - 1$ edges).

Pf.

- Consider a cheapest $v \rightsquigarrow t$ path P that uses the fewest number of edges.
- If P contains a cycle W, can remove portion of P corresponding to W without increasing the cost. ▪

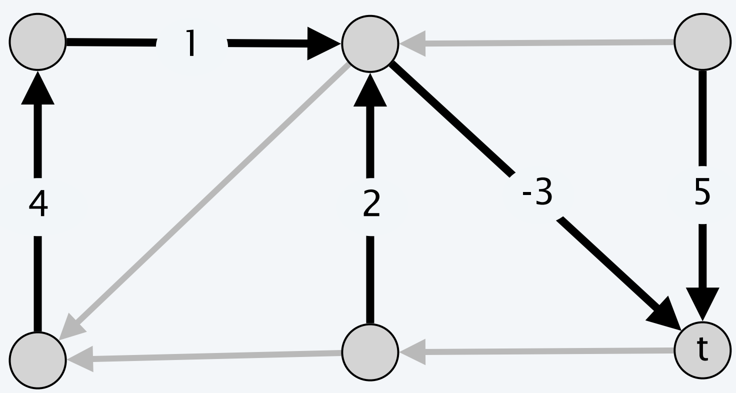


c(W) ≥ 0

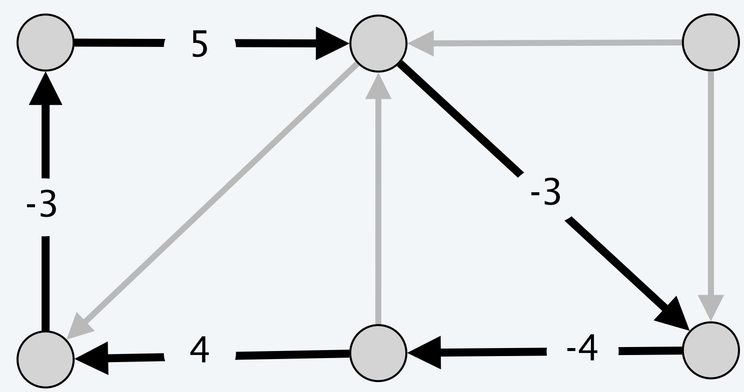
Shortest path and negative cycle problems

Shortest path problem. Given a digraph $G = (V, E)$ with edge weights c_{vw} and no negative cycles, find cheapest $v \rightarrow t$ path for each node v.

Negative cycle problem. Given a digraph $G = (V, E)$ with edge weights c_{vw}, find a negative cycle (if one exists).



shortest-paths tree



negative cycle

Shortest paths: dynamic programming

Def. $OPT(i, v)$ = cost of shortest $v \rightsquigarrow t$ path that uses $\leq i$ edges.

- Case 1: Cheapest $v \rightsquigarrow t$ path uses $\leq i - 1$ edges.
  - $OPT(i, v) = \mathrm{OPT}(i - 1, v)$

optimal substructure property
(proof via exchange argument)

- Case 2: Cheapest $v \rightsquigarrow t$ path uses exactly i edges.
  - if (v, w) is first edge, then OPT uses (v, w), and then selects best $w \rightsquigarrow t$ path using $\leq i - 1$ edges

$$OPT(i, v) = \begin{cases} \infty & \text{if } i = 0 \\ \min\left\{ OPT(i-1, v),\ \min\limits_{(v,w) \in E} \{ OPT(i-1, w) + c_{vw} \} \right\} & \text{otherwise} \end{cases}$$

Observation. If no negative cycles, $OPT(n - 1, v)$ = cost of cheapest $v \rightsquigarrow t$ path.

Pf. By Lemma 2, cheapest $v \rightsquigarrow t$ path is simple. ▪

Shortest paths: implementation

SHORTEST-PATHS (V, E, c, t)

FOREACH node $v \in V$

  $M[0, v] \leftarrow \infty$.

$M[0, t] \leftarrow 0$.

FOR i = 1 TO $n - 1$

  FOREACH node $v \in V$

    $M[i, v] \leftarrow M[i-1, v]$.

    FOREACH edge $(v, w) \in E$

      $M[i, v] \leftarrow \min \{ M[i, v],\ M[i-1, w] + c_{vw} \}$.

Shortest paths: implementation

Theorem 1. Given a digraph $G = (V, E)$ with no negative cycles, the dynamic programming algorithm computes the cost of the cheapest $v \leadsto t$ path for each node v in $\Theta(mn)$ time and $\Theta(n^2)$ space.

Pf.

- Table requires $\Theta(n^2)$ space.
- Each iteration i takes $\Theta(m)$ time since we examine each edge once. ▪

Finding the shortest paths.

- Approach 1: Maintain a $successor(i, v)$ that points to next node on cheapest $v \leadsto t$ path using at most i edges.
- Approach 2: Compute optimal costs $M[i, v]$ and consider only edges with $M[i, v] = M[i-1, w] + c_{vw}$.

Shortest paths: practical improvements

Space optimization. Maintain two 1d arrays (instead of 2d array).

- $d(v)$ = cost of cheapest $v \rightsquigarrow t$ path that we have found so far.
- $successor(v)$ = next node on a $v \rightsquigarrow t$ path.

Performance optimization. If $d(w)$ was not updated in iteration $i - 1$, then no reason to consider edges entering w in iteration i.

Bellman-Ford: efficient implementation

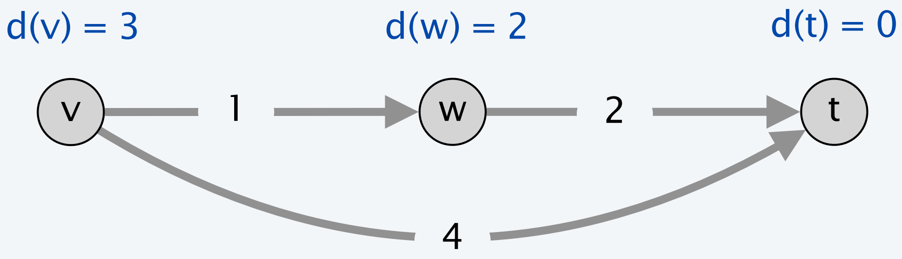
```
BELLMAN-FORD (V, E, c, t)
_______________________________________________
FOREACH node v ∈ V
    d(v) ← ∞.
    successor(v) ← null.
d(t) ← 0.
FOR i = 1 TO n – 1
    FOREACH node w ∈ V
        IF (d(w) was updated in previous iteration)
            FOREACH edge (v, w) ∈ E
                IF ( d(v) > d(w) + c_vw)
                    d(v) ← d(w) + c_vw.
                    successor(v) ← w.
    IF no d(w) value changed in iteration i, STOP.
_______________________________________________
```

1 pass

Bellman-Ford: analysis

Claim. After ~~the i^{th} pass of Bellman-Ford, $d(v)$ equals the cost of the cheapest $v \rightsquigarrow t$ path using at most i edges~~.

Counterexample. Claim is false!



if nodes w considered before node v, then d(v) = 3 after 1 pass

Bellman-Ford: analysis

Lemma 3. Throughout Bellman-Ford algorithm, $d(v)$ is the cost of some $v \rightsquigarrow t$ path; after the i^{th} pass, $d(v)$ is no larger than the cost of the cheapest $v \rightsquigarrow t$ path using $\leq i$ edges.

Pf. [by induction on i]

- Assume true after i^{th} pass.
- Let P be any $v \rightsquigarrow t$ path with $i + 1$ edges.
- Let (v, w) be first edge on path and let P' be subpath from w to t.
- By inductive hypothesis, $d(w) \leq c(P')$ since P' is a $w \rightsquigarrow t$ path with i edges.
- After considering v in pass i+1: $d(v) \leq c_{vw} + d(w)$

$$\begin{aligned} d(v) &\leq c_{vw} + d(w) \\ &\leq c_{vw} + c(P') \\ &= c(P) \quad \blacksquare \end{aligned}$$

Theorem 2. Given a digraph with no negative cycles, Bellman-Ford computes the costs of the cheapest $v \rightsquigarrow t$ paths in $O(mn)$ time and $\Theta(n)$ extra space.

Pf. Lemmas 2 + 3. ▪

can be substantially
faster in practice

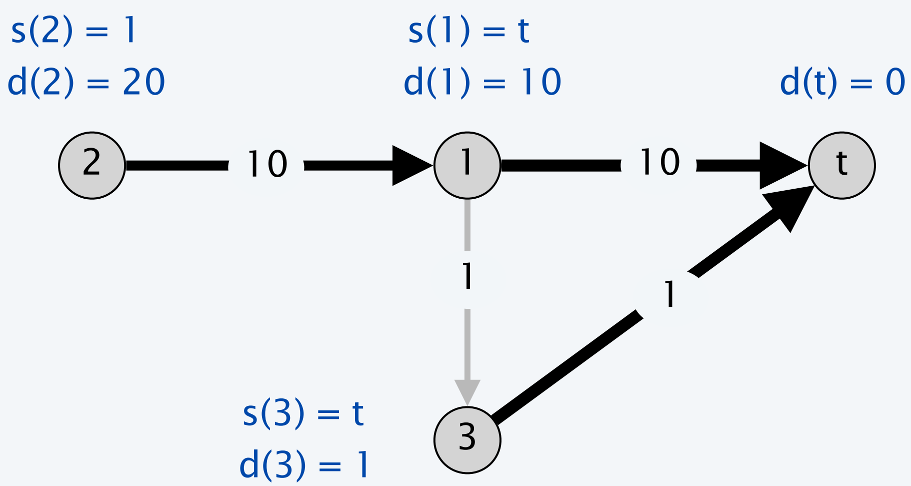
Bellman-Ford: analysis

Claim. ~~Throughout the Bellman-Ford algorithm, following *successor*(v) pointers gives a directed path from v to t of cost $d(v)$.~~

Counterexample. Claim is false!

- Cost of successor $v \rightarrow t$ path may have strictly lower cost than $d(v)$.

consider nodes in order: t, 1, 2, 3



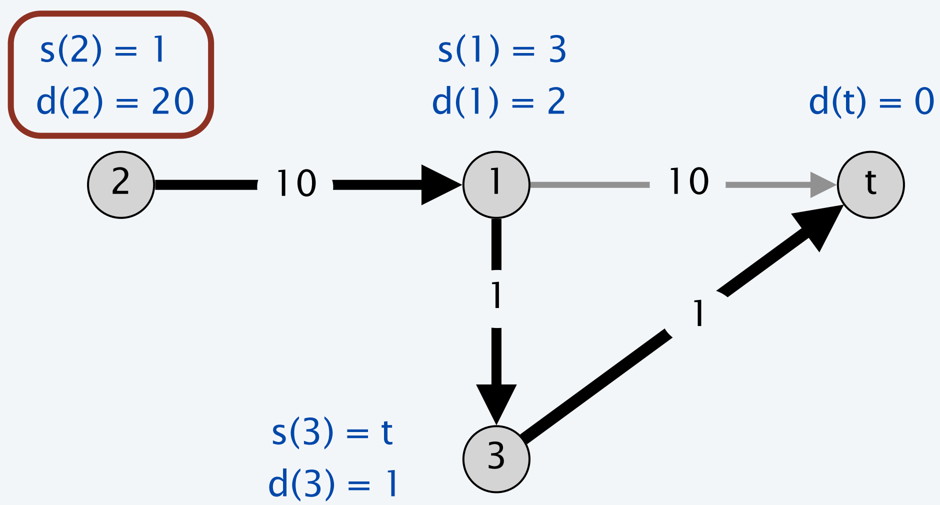
Bellman-Ford: analysis

Claim. ~~Throughout the Bellman-Ford algorithm, following $successor(v)$ pointers gives a directed path from v to t of cost $d(v)$.~~

Counterexample. Claim is false!

- Cost of successor $v \rightarrow t$ path may have strictly lower cost than $d(v)$.

consider nodes in order: t, 1, 2, 3



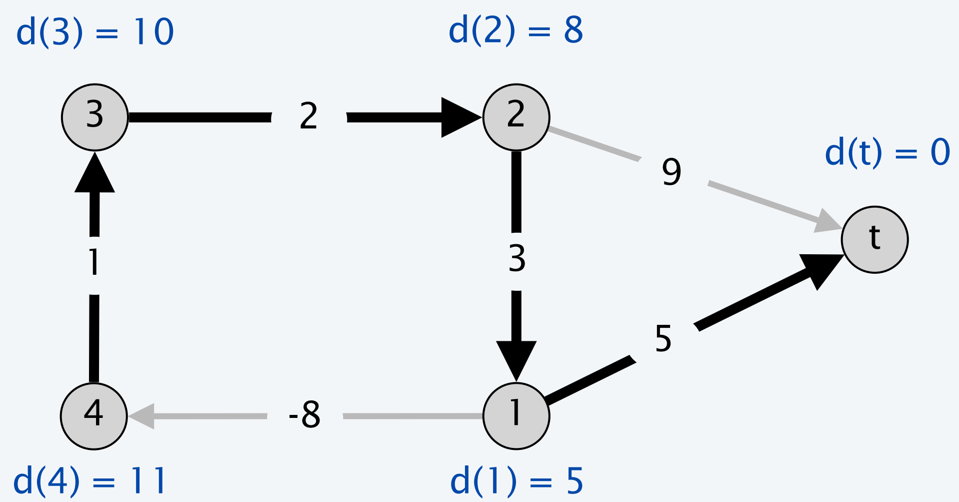
Bellman-Ford: analysis

Claim. ~~Throughout the Bellman-Ford algorithm, following *successor*(*v*) pointers gives a directed path from *v* to *t* of cost *d*(*v*).~~

Counterexample. Claim is false!

- Cost of successor $v \rightarrow t$ path may have strictly lower cost than $d(v)$.
- Successor graph may have cycles.

consider nodes in order: t, 1, 2, 3, 4



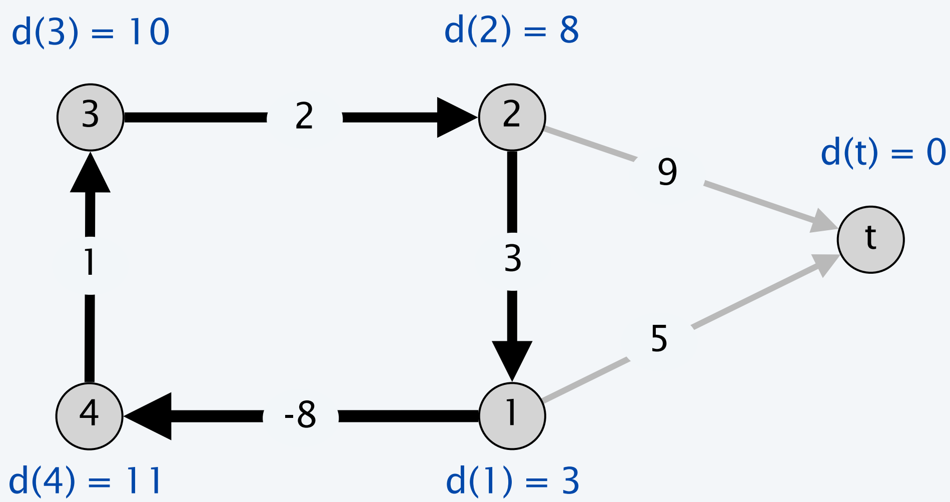
Bellman-Ford: analysis

Claim. ~~Throughout the Bellman-Ford algorithm, following $successor(v)$ pointers gives a directed path from v to t of cost $d(v)$.~~

Counterexample. Claim is false!

- Cost of successor $v \rightarrow t$ path may have strictly lower cost than $d(v)$.
- Successor graph may have cycles.

consider nodes in order: t, 1, 2, 3, 4



Bellman-Ford: finding the shortest path

Lemma 4. If the successor graph contains a directed cycle W, then W is a negative cycle.

Pf.

- If $successor(v) = w$, we must have $d(v) \geq d(w) + c_{vw}$.
  (LHS and RHS are equal when $successor(v)$ is set; $d(w)$ can only decrease; $d(v)$ decreases only when $successor(v)$ is reset)
- Let $v_1 \to v_2 \to \ldots \to v_k$ be the nodes along the cycle W.
- Assume that (v_k, v_1) is the last edge added to the successor graph.
- Just prior to that:

$$\begin{array}{lclcl} d(v_1) & \geq & d(v_2) & + & c(v_1, v_2) \\ d(v_2) & \geq & d(v_3) & + & c(v_2, v_3) \\ \vdots & & \vdots & & \vdots \\ d(v_{k-1}) & \geq & d(v_k) & + & c(v_{k-1}, v_k) \\ d(v_k) & > & d(v_1) & + & c(v_k, v_1) \end{array}$$

holds with strict inequality since we are updating $d(v_k)$

- Adding inequalities yields $\underbrace{c(v_1, v_2) + c(v_2, v_3) + \ldots + c(v_{k-1}, v_k) + c(v_k, v_1) < 0}_{\text{W is a negative cycle}}$. ▪

Bellman-Ford: finding the shortest path

Theorem 3. Given a digraph with no negative cycles, Bellman-Ford finds the cheapest $s \rightsquigarrow t$ paths in $O(mn)$ time and $\Theta(n)$ extra space.

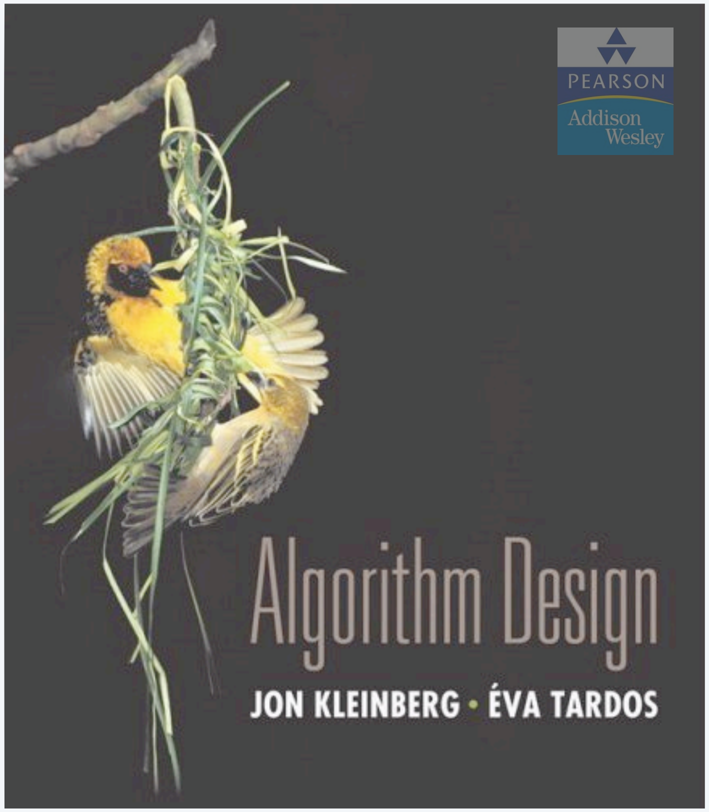
Pf.

- The successor graph cannot have a negative cycle. [Lemma 4]
- Thus, following the successor pointers from s yields a directed path to t.
- Let $s = v_1 \to v_2 \to \ldots \to v_k = t$ be the nodes along this path P.
- Upon termination, if $successor(v) = w$, we must have $d(v) = d(w) + c_{vw}$.
  (LHS and RHS are equal when $successor(v)$ is set; $d(\cdot)$ did not change)

  since algorithm terminated
- Thus,

$$
\begin{array}{rcll}
d(v_1) & = & d(v_2) & + \; c(v_1, v_2) \\
d(v_2) & = & d(v_3) & + \; c(v_2, v_3) \\
\vdots & \vdots & & \vdots \\
d(v_{k-1}) & = & d(v_k) & + \; c(v_{k-1}, v_k)
\end{array}
$$

- Adding equations yields $\underbrace{d(s)}_{\substack{\text{min cost} \\ \text{of any } s \rightsquigarrow t \text{ path} \\ \text{(Theorem 2)}}} = \underbrace{d(t)}_{0} + \underbrace{c(v_1, v_2) + c(v_2, v_3) + \ldots + c(v_{k-1}, v_k)}_{\text{cost of path P}}$. ▪



SECTION 6.9

6. DYNAMIC PROGRAMMING II

- *sequence alignment*
- *Hirschberg's algorithm*
- *Bellman-Ford*
- *distance vector protocols*
- *negative cycles in a digraph*

Distance vector protocols

Communication network.
- Node ≈ router.
- Edge ≈ direct communication link.
- Cost of edge ≈ delay on link. ← naturally nonnegative, but Bellman-Ford used anyway!

Dijkstra's algorithm. Requires global information of network.

Bellman-Ford. Uses only local knowledge of neighboring nodes.

Synchronization. We don't expect routers to run in lockstep. The order in which each `foreach` loop executes in not important. Moreover, algorithm still converges even if updates are asynchronous.

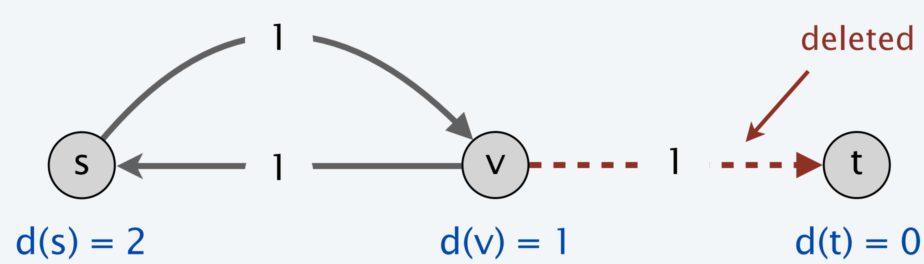
Distance vector protocols

Distance vector protocols. ["routing by rumor"]

- Each router maintains a vector of shortest path lengths to every other node (distances) and the first hop on each path (directions).
- Algorithm: each router performs n separate computations, one for each potential destination node.

Ex. RIP, Xerox XNS RIP, Novell's IPX RIP, Cisco's IGRP, DEC's DNA Phase IV, AppleTalk's RTMP.

Caveat. Edge costs may change during algorithm (or fail completely).



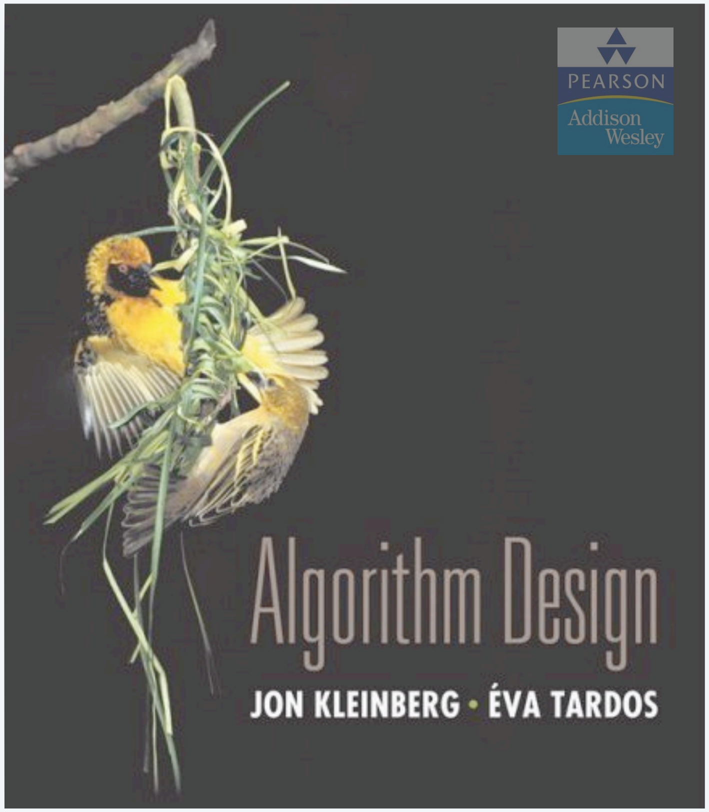
"counting to infinity"

Path vector protocols

Link state routing.

- Each router also stores the entire path. not just the distance and first hop
- Based on Dijkstra's algorithm.
- Avoids "counting-to-infinity" problem and related difficulties.
- Requires significantly more storage.

Ex. Border Gateway Protocol (BGP), Open Shortest Path First (OSPF).



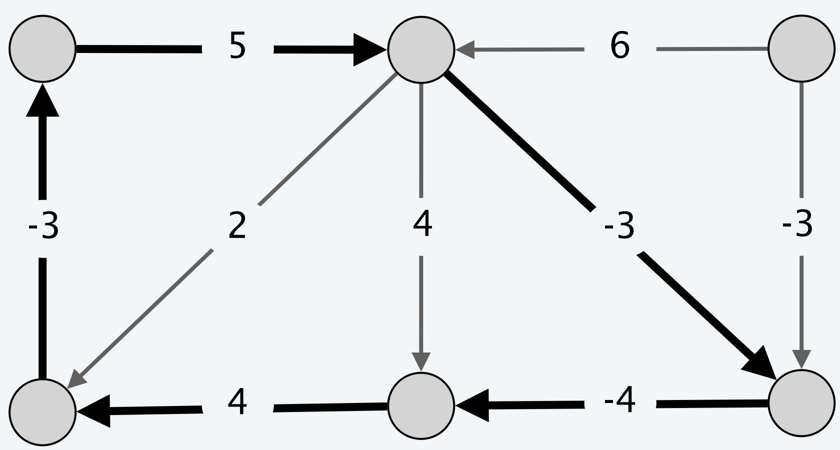
SECTION 6.10

6. DYNAMIC PROGRAMMING II

- *sequence alignment*
- *Hirschberg's algorithm*
- *Bellman-Ford*
- *distance vector protocol*
- *negative cycles in a digraph*

Detecting negative cycles

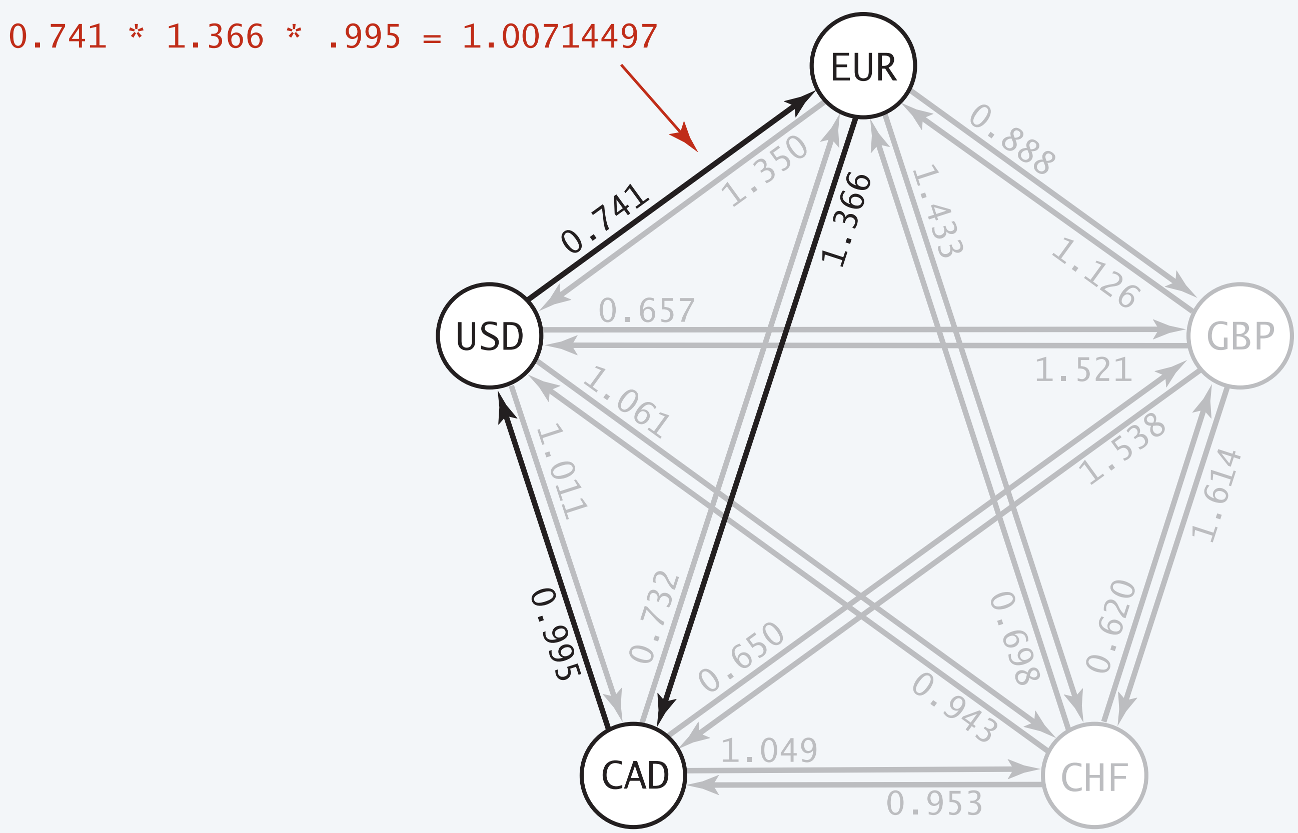
Negative cycle detection problem. Given a digraph $G = (V, E)$, with edge weights c_{vw}, find a negative cycle (if one exists).



Detecting negative cycles: application

Currency conversion. Given n currencies and exchange rates between pairs of currencies, is there an arbitrage opportunity?

Remark. Fastest algorithm very valuable!



Detecting negative cycles

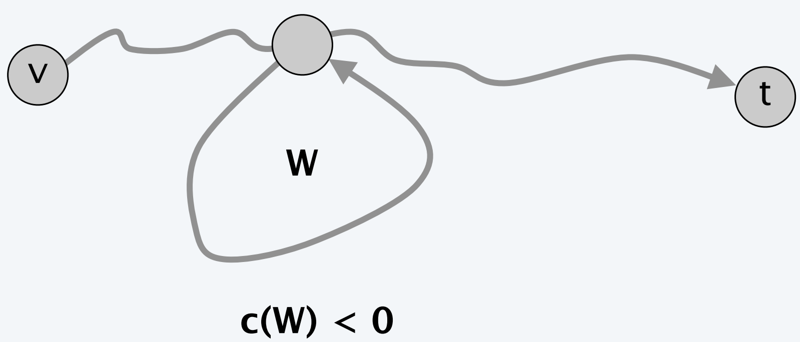
Lemma 5. If $OPT(n, v) = \text{OPT}(n-1, v)$ for all v, then no negative cycle can reach t.

Pf. Bellman-Ford algorithm. ▪

Lemma 6. If $OPT(n, v) < \text{OPT}(n-1, v)$ for some node v, then (any) cheapest path from v to t contains a cycle W. Moreover W is a negative cycle.

Pf. [by contradiction]

- Since $OPT(n, v) < \text{OPT}(n-1, v)$, we know that shortest $v \rightsquigarrow t$ path P has exactly n edges.
- By pigeonhole principle, P must contain a directed cycle W.
- Deleting W yields a $v \rightsquigarrow t$ path with $< n$ edges $\Rightarrow$ W has negative cost. ▪



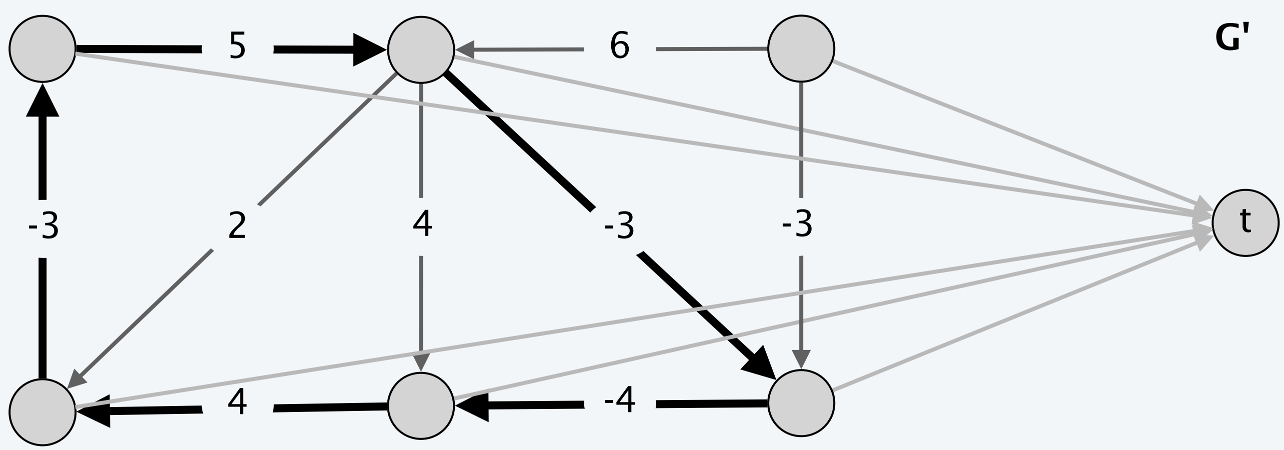
Detecting negative cycles

Theorem 4. Can find a negative cycle in $\Theta(mn)$ time and $\Theta(n^2)$ space.

Pf.

- Add new node t and connect all nodes to t with 0-cost edge.
- G has a negative cycle iff G' has a negative cycle than can reach t.
- If $OPT(n, v) = OPT(n-1, v)$ for all nodes v, then no negative cycles.
- If not, then extract directed cycle from path from v to t.

  (cycle cannot contain t since no edges leave t) ▪



Detecting negative cycles

Theorem 5. Can find a negative cycle in $O(mn)$ time and $O(n)$ extra space.

Pf.

- Run Bellman-Ford for n passes (instead of $n - 1$) on modified digraph.
- If no $d(v)$ values updated in pass n, then no negative cycles.
- Otherwise, suppose $d(s)$ updated in pass n.
- Define $pass(v)$ = last pass in which $d(v)$ was updated.
- Observe $pass(s) = n$ and $pass(successor(v)) \geq pass(v) - 1$ for each v.
- Following successor pointers, we must eventually repeat a node.
- Lemma 4 $\Rightarrow$ this cycle is a negative cycle. ▪

Remark. See p. 304 for improved version and early termination rule.
(Tarjan's subtree disassembly trick)