Solution

i	1					2					3
j	1	2	3	4	5	1	2	3	4	5	
found	no			yes		no					
answer	$A \subseteq B$									$A \not\subseteq B$	

■

In the exercises at the end of this section, you are asked to write an algorithm to check whether a given element is in a given set. To do this, you can represent the set as a one-dimensional array and compare the given element with successive elements of the array to determine whether the two elements are equal. If they are, then the element is in the set; if the given element does not equal any element of the array, then the element is not in the set.

Exercise Set 5.1*

1. Which of the following sets are equal?

$$A = \{a, b, c, d\} \qquad B = \{d, e, a, c\}$$
$$C = \{d, b, a, c\} \qquad D = \{a, a, d, e, c, e\}$$

2. Is $4 = \{4\}$? Explain.

3. Which of the following sets are equal?

$$A = \{0, 1, 2\}$$
$$B = \{x \in \mathbf{R} \mid -1 \le x < 3\}$$
$$C = \{x \in \mathbf{R} \mid -1 < x < 3\}$$
$$D = \{x \in \mathbf{Z} \mid -1 < x < 3\}$$
$$E = \{x \in \mathbf{Z}^+ \mid -1 < x < 3\}$$

4. Indicate the elements in each set defined in (a)–(f).
   a. $S = \{n \in \mathbf{Z} \mid n = (-1)^k$, for some integer $k\}$.
   b. $T = \{m \in \mathbf{Z} \mid m = 1 + (-1)^i$, for some integer $i\}$.
   c. $U = \{r \in \mathbf{Z} \mid 2 \le r \le -2\}$
   d. $V = \{s \in \mathbf{Z} \mid s > 2 \text{ or } s < 3\}$
   e. $W = \{t \in \mathbf{Z} \mid -1 < t < -3\}$
   f. $X = \{u \in \mathbf{Z} \mid u \le 4 \text{ or } u \ge 1\}$

5. **a.** Is the number 0 in $\emptyset$? Why? **b.** Is $\emptyset = \{\emptyset\}$? Why?
   c. Is $\emptyset \in \{\emptyset\}$? Why? d. Is $\emptyset \in \emptyset$? Why?

6. Write in words how to read each of the following out loud. Then write the shorthand notation for each set.
   a. $\{x \in U \mid x \in A \text{ and } x \in B\}$
   b. $\{x \in U \mid x \in A \text{ or } x \in B\}$
   c. $\{x \in U \mid x \in A \text{ and } x \notin B\}$
   d. $\{x \in U \mid x \notin A\}$

7. Let $A = \{c, d, f, g\}$, $B = \{f, j\}$, and $C = \{d, g\}$. Answer each of the following questions. Give reasons for your answers.
   a. Is $B \subseteq A$? b. Is $C \subseteq A$?
   c. Is $C \subseteq C$? **d.** Is C a proper subset of A?

8. **a.** Is $3 \in \{1, 2, 3\}$? **b.** Is $1 \subseteq \{1\}$?
   c. Is $\{2\} \in \{1, 2\}$? d. Is $\{3\} \in \{1, \{2\}, \{3\}\}$?
   e. Is $1 \in \{1\}$? **f.** Is $\{2\} \subseteq \{1, \{2\}, \{3\}\}$?
   g. Is $\{1\} \subseteq \{1, 2\}$? h. Is $1 \in \{\{1\}, 2\}$?
   i. Is $\{1\} \subseteq \{1, \{2\}\}$? j. Is $\{1\} \subseteq \{1\}$?

9. Let $A = \{1, 3, 5, 7, 9\}$, $B = \{3, 6, 9\}$, and $C = \{2, 4, 6, 8\}$. Find each of the following:
   a. $A \cup B$ **b.** $A \cap B$ **c.** $A \cup C$ **d.** $A \cap C$
   e. $A - B$ f. $B - A$ g. $B \cup C$ h. $B \cap C$

10. Let the universal set be the set $\mathbf{R}$ of all real numbers and let $A = \{x \in \mathbf{R} \mid 0 < x \le 2\}$, $B = \{x \in \mathbf{R} \mid 1 \le x < 4\}$ and $C = \{x \in \mathbf{R} \mid 3 \le x < 9\}$. Find each of the following:
    a. $A \cup B$ b. $A \cap B$ c. A^c d. $A \cup C$
    e. $A \cap C$ f. B^c g. $A^c \cap B^c$
    h. $A^c \cup B^c$ i. $(A \cap B)^c$ j. $(A \cup B)^c$

11. Let the universal set be the set $\mathbf{R}$ of all real numbers and let $A = \{x \in \mathbf{R} \mid -3 \le x \le 0\}$, $B = \{x \in \mathbf{R} \mid -1 < x < 2\}$, and $C = \{x \in \mathbf{R} \mid 6 < x \le 8\}$. Find each of the following:
    a. $A \cup B$ b. $A \cap B$ c. A^c d. $A \cup C$
    e. $A \cap C$ f. B^c g. $A^c \cap B^c$
    h. $A^c \cup B^c$ i. $(A \cap B)^c$ j. $(A \cup B)^c$

12. Indicate which of the following relationships are true and which are false:
    a. $\mathbf{Z}^+ \subseteq \mathbf{Q}$ **b.** $\mathbf{R}^- \subseteq \mathbf{Q}$
    c. $\mathbf{Q} \subseteq \mathbf{Z}$ **d.** $\mathbf{Z}^- \cup \mathbf{Z}^+ = \mathbf{Z}$
    e. $\mathbf{Z}^- \cap \mathbf{Z}^+ = \emptyset$ f. $\mathbf{Q} \cap \mathbf{R} = \mathbf{Q}$
    g. $\mathbf{Q} \cup \mathbf{Z} = \mathbf{Q}$ h. $\mathbf{Z}^+ \cap \mathbf{R} = \mathbf{Z}^+$
    i. $\mathbf{Z} \cup \mathbf{Q} = \mathbf{Z}$

*For exercises with blue numbers or letters, solutions are given in Appendix B. The symbol ***H*** indicates that only a hint or a partial solution is given. The symbol ✱ signals that an exercise is more challenging than usual.

13. **a.** Write a negation for the following statement: $\forall$ sets A, if $A \subseteq \mathbf{R}$ then $A \subseteq \mathbf{Z}$. Which is true, the statement or its negation? Explain.
    b. Write a negation for the following statement: $\forall$ sets S, if $S \subseteq Q^+$ then $S \subseteq Q^-$. Which is true, the statement or its negation? Explain.

14. Let sets R, S, and T be defined as follows:

$$R = \{x \in \mathbf{Z} \mid x \text{ is divisible by } 2\}$$
$$S = \{y \in \mathbf{Z} \mid y \text{ is divisible by } 3\}$$
$$T = \{z \in \mathbf{Z} \mid z \text{ is divisible by } 6\}$$

    a. Is $R \subseteq T$? Explain.
    b. Is $T \subseteq R$? Explain.
    c. Is $T \subseteq S$? Explain.
    d. Find $R \cap S$. Explain.

15. Let $A = \{n \in \mathbf{Z} \mid n = 5r \text{ for some integer } r\}$ and $B = \{m \in \mathbf{Z} \mid m = 20s \text{ for some integer } s\}$.
    a. Is $A \subseteq B$?
    b. Is $B \subseteq A$?

16. Let $C = \{n \in \mathbf{Z} \mid n = 6r - 5 \text{ for some integer } r\}$ and $D = \{m \in \mathbf{Z} \mid m = 3s + 1 \text{ for some integer } s\}$.
    a. Is $C \subseteq D$?
    b. Is $D \subseteq C$?

17. Let $A = \{m \in \mathbf{Z} \mid m = 5i - 1, \text{ for some integer } i\}$, $B = \{n \in \mathbf{Z} \mid n = 3j + 2, \text{ for some integer } j\}$, $C = \{p \in \mathbf{Z} \mid p = 5r + 4, \text{ for some integer } r\}$, and $D = \{q \in \mathbf{Z} \mid q = 3s - 1, \text{ for some integer } s\}$.
    a. Is $A = B$? Explain.
    b. Is $A = C$? Explain.
    c. Is $A = D$? Explain.
    d. Is $B = D$? Explain.

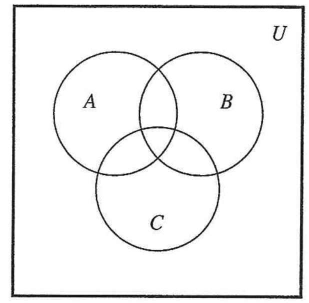
18. In each of the following, draw a Venn diagram for sets A, B, and C that satisfy the given conditions:
    a. $A \subseteq B$; $C \subseteq B$; $A \cap C = \emptyset$
    b. $C \subseteq A$; $B \cap C = \emptyset$

19. Draw Venn diagrams to describe sets A, B, and C that satisfy the given conditions.
    a. $A \cap B = \emptyset, A \subseteq C, C \cap B \neq \emptyset$
    b. $A \subseteq B, C \subseteq B, A \cap C \neq \emptyset$
    c. $A \cap B \neq \emptyset, B \cap C \neq \emptyset, A \cap C = \emptyset, A \not\subseteq B, C \not\subseteq B$

20. Let $A = \{a, b, c\}$, $B = \{b, c, d\}$, and $C = \{b, c, e\}$.
    a. Find $A \cup (B \cap C)$, $(A \cup B) \cap C$, and $(A \cup B) \cap (A \cup C)$. Which of these sets are equal?
    b. Find $A \cap (B \cup C)$, $(A \cap B) \cup C$, and $(A \cap B) \cup (A \cap C)$. Which of these sets are equal?
    c. Find $(A - B) - C$ and $A - (B - C)$. Are these sets equal?

21. Consider the Venn diagram shown in the next column. For each of (a)–(f), copy the diagram and shade the region corresponding to the indicated set.

a. $A \cap B$	b. $B \cup C$	c. A^c
d. $A - (B \cup C)$	e. $(A \cup B)^c$	f. $A^c \cap B^c$



22. **a.** Is $\{\{a, d, e\}, \{b, c\}, \{d, f\}\}$ a partition of $\{a, b, c, d, e, f\}$?
    b. Is $\{\{w, x, v\}, \{u, y, q\}, \{p, z\}\}$ a partition of $\{p, q, u, v, w, x, y, z\}$?
    c. Is $\{\{5, 4\}, \{7, 2\}, \{1, 3, 4\}, \{6, 8\}\}$ a partition of $\{1, 2, 3, 4, 5, 6, 7, 8\}$?
    d. Is $\{\{3, 7, 8\}, \{2, 9\}, \{1, 4, 5\}\}$ a partition of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$?
    e. Is $\{\{1, 5\}, \{4, 7\}, \{2, 8, 6, 3\}\}$ a partition of $\{1, 2, 3, 4, 5, 6, 7, 8\}$?

23. Let E be the set of all even integers and O the set of all odd integers. Is $\{E, O\}$ a partition of $\mathbf{Z}$, the set of all integers? Explain your answer.

24. Let $\mathbf{R}$ be the set of all real numbers. Is $\{\mathbf{R}^+, \mathbf{R}^-, \{0\}\}$ a partition of $\mathbf{R}$? Explain your answer.

25. Let $\mathbf{Z}$ be the set of all integers and let

$$A_0 = \{n \in \mathbf{Z} \mid n = 4k, \text{ for some integer } k\},$$
$$A_1 = \{n \in \mathbf{Z} \mid n = 4k + 1, \text{ for some integer } k\},$$
$$A_2 = \{n \in \mathbf{Z} \mid n = 4k + 2, \text{ for some integer } k\}, \text{ and}$$
$$A_3 = \{n \in \mathbf{Z} \mid n = 4k + 3, \text{ for some integer } k\}.$$

    Is $\{A_0, A_1, A_2, A_3\}$ a partition of $\mathbf{Z}$? Explain your answer.

26. Suppose $A = \{1, 2\}$ and $B = \{2, 3\}$. Find each of the following:
    a. $\mathscr{P}(A \cap B)$ b. $\mathscr{P}(A)$
    c. $\mathscr{P}(A \cup B)$ d. $\mathscr{P}(A \times B)$

27. **a.** Suppose $A = \{1\}$ and $B = \{u, v\}$. Find $\mathscr{P}(A \times B)$.
    b. Suppose $X = \{a, b\}$ and $Y = \{x, y\}$. Find $\mathscr{P}(X \times Y)$.

28. a. Find $\mathscr{P}(\emptyset)$. **b.** Find $\mathscr{P}(\mathscr{P}(\emptyset))$.
    c. Find $\mathscr{P}(\mathscr{P}(\mathscr{P}(\emptyset)))$.

29. Let $A = \{x, y, z, w\}$ and $B = \{a, b\}$. List the elements of each of the following sets:
    a. $A \times B$ b. $B \times A$
    c. $A \times A$ d. $B \times B$

30. Let $A = \{1, 2, 3\}$, $B = \{u, v\}$, and $C = \{m, n\}$. List the elements of each of the following sets:
    a. $A \times (B \times C)$ b. $(A \times B) \times C$ c. $A \times B \times C$

31. Trace the action of Algorithm 5.1.1 on the variables i, j, *found*, and *answer* for $m = 3, n = 3$, and sets A and B represented as the arrays $a[1] = u, a[2] = v, a[3] = w$, $b[1] = w, b[2] = u$, and $b[3] = v$.

32. Trace the action of Algorithm 5.1.1 on the variables i, j, *found*, and *answer* for $m = 4, n = 4$, and sets A and B represented as the arrays $a[1] = u, a[2] = v, a[3] = w$, $a[4] = x, b[1] = r, b[2] = u, b[3] = y, b[4] = z$.

33. Write an algorithm to determine whether a given element x belongs to a given set, which is represented as an array $a[1], a[2], \ldots, a[n]$.

5.2 Properties of Sets

... only the last line is a genuine theorem here—everything else is in the fantasy.
— Douglas Hofstadter, *Gödel, Escher, Bach*, 1979

It is possible to list many relations involving unions, intersections, complements, and differences of sets. Some of these are true for all sets, whereas others fail to hold in some cases. In this section we show how to establish basic set properties using *element arguments,* the most basic method used for proofs involving sets, and we discuss a variation used to prove that a set is empty. In the next section we will show how to disprove a proposed set property by constructing a counterexample and how to use an algebraic technique to derive new set properties from set properties already known to be true.

We begin by listing some set properties that involve subset relations. As you read them, keep in mind that the operations of union, intersection, and difference take precedence over set inclusion. Thus, for example, $A \cap B \subseteq C$ means $(A \cap B) \subseteq C$.

Theorem 5.2.1 Some Subset Relations

1. *Inclusion of Intersection:* For all sets A and B,
$$\text{(a) } A \cap B \subseteq A \quad \text{and} \quad \text{(b) } A \cap B \subseteq B.$$
2. *Inclusion in Union:* For all sets A and B,
$$\text{(a) } A \subseteq A \cup B \quad \text{and} \quad \text{(b) } B \subseteq A \cup B.$$
3. *Transitive Property of Subsets:* For all sets A, B, and C,
$$\text{if } A \subseteq B \text{ and } B \subseteq C, \text{ then } A \subseteq C.$$

The conclusion of each part of Theorem 5.2.1 states that one set is a subset of another. Recall that by definition of subset, if X and Y are sets, then

$$X \subseteq Y \quad \Leftrightarrow \quad \forall x, \text{if } x \in X \text{ then } x \in Y.$$

Since the definition of subset is a universal conditional statement, the most basic way to prove that one set is a subset of another is as follows:

Element Argument: The Basic Method for Proving That One Set Is a Subset of Another

Let sets X and Y be given. To prove that $X \subseteq Y$,

1. suppose that x is a particular but arbitrarily chosen element of X,
2. show that x is an element of Y.

Example 5.2.5 A Proof for a Conditional Statement

Prove that for all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C^c$, then $A \cap C = \emptyset$.

Solution Since the statement to be proved is both universal and conditional, you start with the method of direct proof:

Suppose A, B, and C are arbitrarily chosen sets that satisfy the condition: $A \subseteq B$ and $B \subseteq C^c$.

Show that $A \cap C = \emptyset$.

Since the conclusion to be shown is that a certain set is empty, you can use the principle for proving that a set equals the empty set. A complete proof is shown below.

Proposition 5.2.6

For all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C^c$, then $A \cap C = \emptyset$.

Proof:

Suppose A, B, and C are any sets such that $A \subseteq B$ and $B \subseteq C^c$. We must show that $A \cap C = \emptyset$. Suppose not. That is, suppose there is an element x in $A \cap C$. By definition of intersection, $x \in A$ and $x \in C$. Then, since $A \subseteq B$, $x \in B$ by definition of subset. Also, since $B \subseteq C^c$, then $x \in C^c$ by definition of subset again. It follows by definition of complement that $x \notin C$. Thus $x \in C$ and $x \notin C$, which is a contradiction. So the supposition that there is an element x in $A \cap C$ is false, and thus $A \cap C = \emptyset$ *[as was to be shown]*.

■

Exercise Set 5.2

1. **a.** To say that an element is in $A \cap (B \cup C)$ means that it is in ____(1)____ and in ____(2)____.
   b. To say that an element is in $(A \cap B) \cup C$ means that it is in ____(1)____ or in ____(2)____.
   c. To say that an element is in $A - (B \cap C)$ means that it is in ____(1)____ and not in ____(2)____.

2. The following are two proofs that for all sets A and B, $A - B \subseteq A$. The first is less formal, and the second is more formal. Fill in the blanks.
   a. **Proof:** Suppose A and B are any sets. To show that $A - B \subseteq A$, we must show that every element in ____(1)____ is in ____(2)____. But any element in $A - B$ is in ____(3)____ and not in ____(4)____ (by definition of $A - B$). In particular, such an element is in A.
   b. **Proof:** Suppose A and B are any sets and $x \in A - B$. *[We must show that* ____(1)____.*]* By definition of set difference, $x \in$ ____(2)____ and $x \notin$ ____(3)____. In particular, $x \in$ ____(4)____ *[which is what was to be shown]*.

3. The following is a proof that for all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. Fill in the blanks.
   Proof: Suppose A, B, and C are sets and $A \subseteq B$ and $B \subseteq C$. To show that $A \subseteq C$, we must show that every element in ____(1)____ is in ____(2)____. But given any element in A, that element is in ____(3)____ (because $A \subseteq B$), and so that element is also in ____(4)____ (because ____(5)____). Hence $A \subseteq C$.

4. The following is a proof that for all sets A and B, if $A \subseteq B$, then $A \cup B \subseteq B$. Fill in the blanks.
   Proof: Suppose A and B are any sets and $A \subseteq B$. *[We must show that* ____(a)____.*]* Let $x \in$ ____(b)____. *[We must show that* ____(c)____.*]* By definition of union, $x \in$ ____(d)____ ____(e)____ $x \in$ ____(f)____. In case $x \in$ ____(g)____, then since $A \subseteq B$, $x \in$ ____(h)____. In case $x \in B$, then clearly $x \in B$. So in either case, $x \in$ ____(i)____ *[as was to be shown]*.

5. Prove that for all sets A and B, $B - A = B \cap A^c$.

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6. The following is a proof that for any sets A, B, and C, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Fill in the blanks.

Proof: Suppose A, B, and C are any sets.

(1) $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$:

Let $x \in A \cap (B \cup C)$. [*We must show that* $x \in \underline{\text{(a)}}$.] By definition of intersection, $x \in \underline{\text{(b)}}$ and $x \in \underline{\text{(c)}}$. Thus $x \in A$ and, by definition of union, $x \in B$ or $\underline{\text{(d)}}$.

***Case 1* ($x \in A$ *and* $x \in B$):** In this case, by definition of intersection, $x \in \underline{\text{(e)}}$, and so, by definition of union, $x \in (A \cap B) \cup (A \cap C)$.

***Case 2* ($x \in A$ *and* $x \in C$):** In this case, $\underline{\text{(f)}}$.

Hence in either case, $x \in (A \cap B) \cup (A \cap C)$ [*as was to be shown*].

[*So* $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ *by definition of subset.*]

(2) $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$:

Let $x \in (A \cap B) \cup (A \cap C)$. [*We must show that* $\underline{\text{(a)}}$.] By definition of union, $x \in \underline{\text{(b)}}$ or $x \in \underline{\text{(c)}}$.

***Case 1* ($x \in A \cap B$):** In this case, by definition of intersection, $\underline{\text{(d)}}$ and $\underline{\text{(e)}}$. Since $x \in B$, then by definition of union, $x \in B \cup C$. Hence $x \in A$ and $x \in B \cup C$, and so, by definition of intersection, $x \in \underline{\text{(f)}}$.

***Case 2* ($x \in A \cap C$):** In this case, $\underline{\text{(g)}}$.

In either case, $x \in A \cap (B \cup C)$ [*as was to be shown*]. [*Thus* $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ *by definition of subset.*]

[*Since both subset relations have been proved, it follows, by definition of set equality, that* $\underline{\text{(h)}}$.]

H **7.** Prove that for all sets A and B, $(A \cap B)^c = A^c \cup B^c$.

Use an element argument to prove each statement in 8–17. Assume that all sets are subsets of a universal set U.

8. For all sets A, B, and C,

$$(A - B) \cup (C - B) = (A \cup C) - B.$$

9. For all sets A, B, and C,

$$(A - B) \cap (C - B) = (A \cap C) - B.$$

H **10.** For all sets A and B, $A \cup (A \cap B) = A$.

11. For all sets A, $A \cup \emptyset = A$.

12. For all sets A, B, and C, if $A \subseteq B$ then $A \cap C \subseteq B \cap C$.

13. For all sets A, B, and C, if $A \subseteq B$ then $A \cup C \subseteq B \cup C$.

14. For all sets A and B, if $A \subseteq B$ then $B^c \subseteq A^c$.

H **15.** For all sets A, B, and C, if $A \subseteq B$ and $A \subseteq C$ then

$$A \subseteq B \cap C.$$

16. For all sets A, B, and C,

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

17. For all sets A, B, and C,

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

18. Find the mistake in the following "proof" that for all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

"**Proof:** Suppose A, B, and C are sets such that $A \subseteq B$ and $B \subseteq C$. Since $A \subseteq B$, there is an element x such that $x \in A$ and $x \in B$. Since $B \subseteq C$, there is an element x such that $x \in B$ and $x \in C$. Hence there is a element x such that $x \in A$ and $x \in C$ and so $A \subseteq C$."

H **19.** Find the mistake in the following "proof."

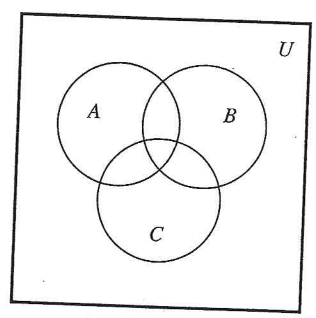
"**Theorem:**" For all sets A and B, $A^c \cup B^c \subseteq (A \cup B)^c$.

"**Proof:** Suppose A and B are sets, and $x \in A^c \cup B^c$. Then $x \in A^c$ or $x \in B^c$ by definition of union. It follows that $x \notin A$ or $x \notin B$ by definition of complement, and so $x \notin A \cup B$ by definition of union. Thus $x \in (A \cup B)^c$ by definition of complement, and hence $A^c \cup B^c \subseteq (A \cup B)^c$."

20. Find the mistake in the following "proof" that for all sets A and B, $(A - B) \cup (A \cap B) \subseteq A$.

"**Proof:** Suppose A and B are sets, and suppose $x \in (A - B) \cup (A \cap B)$. If $x \in A$ then $x \in A - B$. Then, by definition of difference, $x \in A$ and $x \notin B$. Hence $x \in A$, and so $(A - B) \cup (A \cap B) \subseteq A$ by definition of subset."

21. Consider the Venn diagram below.



a. Illustrate one of the distributive laws by shading in the region corresponding to $A \cup (B \cap C)$ on one copy of the diagram and $(A \cup B) \cap (A \cup C)$ on another.

b. Illustrate the other distributive law by shading in the region corresponding to $A \cap (B \cup C)$ on one copy of the diagram and $(A \cap B) \cup (A \cap C)$ on another.

c. Illustrate one of De Morgan's laws by shading in the region corresponding to $(A \cup B)^c$ on one copy of the diagram and $A^c \cap B^c$ on the other. (Leave the set C out of your diagrams.)

d. Illustrate the other De Morgan's law by shading in the region corresponding to $(A \cap B)^c$ on one copy of the diagram and $A^c \cup B^c$ on the other. (Leave the set C out of your diagrams.)

22. Fill in the blanks in the following proof that for all sets A and B, $(A - B) \cap (B - A) = \emptyset$.

Proof: Let A and B be any sets and supppose $(A - B) \cap (B - A) \neq \emptyset$. That is, suppose there were an element x in $\underline{(a)}$. By definition of $\underline{(b)}$, $x \in A - B$ and $x \in \underline{(c)}$. Then by definition of set difference, $x \in A$ and $x \notin B$ and $x \in \underline{(d)}$ and $x \notin \underline{(e)}$. In particular $x \in A$ and $x \notin \underline{(f)}$, which is a contradiction. Hence [*the supposition that* $(A - B) \cap (B - A) \neq \emptyset$ *is false, and so*] $\underline{(g)}$.

Use the element method for proving a set equals the empty set to prove each statement in 23–34. Assume that all sets are subsets of a universal set U.

23. For all sets A and B, $(A - B) \cap (A \cap B) = \emptyset$.

24. For all sets A, B, and C,

$$(A - C) \cap (B - C) \cap (A - B) = \emptyset.$$

25. For all subsets A of a universal set U, $A \cap A^c = \emptyset$.

26. If U denotes a universal set, then $U^c = \emptyset$.

27. For all sets A, $A \times \emptyset = \emptyset$.

28. For all sets A and B, if $A \subseteq B$ then $A \cap B^c = \emptyset$.

29. For all sets A and B, if $B \subseteq A^c$ then $A \cap B = \emptyset$.

30. For all sets A, B, and C, if $A \subseteq B$ and $B \cap C = \emptyset$ then $A \cap C = \emptyset$.

31. For all sets A, B, and C, if $B \subseteq C$ and $A \cap C = \emptyset$, then $A \cap B = \emptyset$.

32. For all sets A, B, and C, if $C \subseteq B - A$, then $A \cap C = \emptyset$.

33. For all sets A, B, and C,

$$\text{if } B \cap C \subseteq A, \text{ then } (C - A) \cap (B - A) = \emptyset.$$

34. For all sets A, B, C, and D,

$$\text{if } A \cap C = \emptyset \text{ then } (A \times B) \cap (C \times D) = \emptyset.$$

Use mathematical induction and the following definitions to prove each statement in 35–37. If n is an integer with $n \geq 3$ and if $C_1, C_2, C_3, \ldots, C_n$ are any sets,

$$C_1 \cup C_2 \cup C_3 \cup \cdots \cup C_n = (C_1 \cup C_2 \cup C_3 \cup \cdots \cup C_{n-1}) \cup C_n,$$

and

$$C_1 \cap C_2 \cap C_3 \cap \cdots \cap C_n = (C_1 \cap C_2 \cap C_3 \cap \cdots \cap C_{n-1}) \cap C_n.$$

(More rigorous versions of the definitions are given in Section 8.4.)

35. *Generalized Distributive Law for Sets:* For any integer $n \geq 1$, if A and $B_1, B_2, B_3, \ldots, B_n$ are any sets, then

$$(A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n) = A \cap (B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_n).$$

36. For any integer $n \geq 1$, if $A_1, A_2, A_3, \ldots, A_n$ and B are any sets, then

$$(A_1 - B) \cup (A_2 - B) \cup \cdots \cup (A_n - B) = (A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n) - B.$$

(To prove the inductive step, you can use the result of exercise 8.)

37. For any integer $n \geq 1$, if $A_1, A_2, A_3, \ldots, A_n$ and B are any sets, then

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) = (A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n) - B.$$

(To prove the inductive step, you can use the result of exercise 9.)

5.3 Disproofs, Algebraic Proofs, and Boolean Algebras

If a fact goes against common sense, and we are nevertheless compelled to accept and deal with this fact, we learn to alter our notion of common sense.
— Phillip J. Davis and Reuben Hersh, *The Mathematical Experience*, 1981

In Section 5.2 we gave examples only of set properties that were true. Occasionally, however, a proposed set property is false. We begin this section by discussing how to disprove such a proposed property. Then we prove an important theorem about the power set of a set and go on to discuss an "algebraic" method for deriving new set properties from set properties already known to be true. We finish the section with an introduction to Boolean algebras.

Disproving an Alleged Set Property

Recall that to show a universal statement is false, it suffices to find one example (called a counterexample) for which it is false.

Theorem 5.3.2(3) Double Complement Law

For all elements a in a Boolean algebra B, $\overline{(\overline{a})} = a$.

Proof:

Suppose B is a Boolean algebra and a is any element of B. Then

$$\begin{aligned} \overline{a} + a &= a + \overline{a} && \text{by the commutative law} \\ &= 1 && \text{by the complement law for } 1 \end{aligned}$$

and

$$\begin{aligned} \overline{a} \cdot a &= a \cdot \overline{a} && \text{by the commutative law} \\ &= 0 && \text{by the complement law for } 0. \end{aligned}$$

Thus a satisfies the two equations with respect to $\overline{a}$ that are satisfied by the complement of $\overline{a}$. From the fact that the complement of a is unique, we conclude that $\overline{(\overline{a})} = a$.

■

Example 5.3.6 Proof of an Idempotent Law

Fill in the blanks in the following proof that for all elements a in a Boolean algebra B, $a + a = a$.

Proof:

Suppose B is a Boolean algebra and a is any element of B. Then

$$\begin{aligned} a &= a + 0 && \underline{\quad (a) \quad} \\ &= a + (a \cdot \overline{a}) && \underline{\quad (b) \quad} \\ &= (a + a) \cdot (a + \overline{a}) && \underline{\quad (c) \quad} \\ &= (a + a) \cdot 1 && \underline{\quad (d) \quad} \\ &= a + a && \underline{\quad (e) \quad}. \end{aligned}$$

Solution

a. because 0 is an identity for +

b. by the complement law for ·

c. by the distributive law for + over ·

d. by the complement law for +

e. because 1 is an identity for ·

■

Exercise Set 5.3

For each of 1–4 find a counterexample to show that the statement is false. Assume all sets are subsets of a universal set U.

1. For all sets A, B, and C, $(A \cap B) \cup C = A \cap (B \cup C)$.

2. For all sets A, B, and C, if $A \subseteq B$ then $A \cap (B \cap C)^c = \emptyset$.

3. For all sets A, B, and C, if $A \not\subseteq B$ and $B \not\subseteq C$ then $A \not\subseteq C$.

4. For all sets A, B, and C, if $B \cap C \subseteq A$ then $(A - B) \cap (A - C) = \emptyset$.

For each of 5–17 prove each statement that is true and find a counterexample for each statement that is false. Assume all sets are subsets of a universal set U.

5. For all sets A, B, and C, $A - (B - C) = (A - B) - C$.

comple-
ude that

algebra B,

and find a
ıme all sets

$B) - C$.

6. For all sets A and B, $A \cap (A \cup B) = A$.

7. For all sets A, B, and C,

$$(A - B) \cap (C - B) = A - (B \cup C).$$

8. For all sets A and B, if $A^c \subseteq B$ then $A \cup B = U$.

9. For all sets A, B, and C, if $A \subseteq C$ and $B \subseteq C$ then $A \cup B \subseteq C$.

10. For all sets A and B, if $A \subseteq B$ then $A \cap B^c = \emptyset$.

H **11.** For all sets A, B, and C, if $A \subseteq B$ and $B \cap C = \emptyset$ then $A \cap C = \emptyset$.

12. For all sets A and B, if $A \cap B = \emptyset$ then $A \times B = \emptyset$.

13. For all sets A and B, if $A \subseteq B$ then $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.

14. For all sets A and B, $\mathscr{P}(A \cup B) \subseteq \mathscr{P}(A) \cup \mathscr{P}(B)$.

H **15.** For all sets A and B, $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$.

16. For all sets A and B, $\mathscr{P}(A \cap B) = \mathscr{P}(A) \cap \mathscr{P}(B)$.

17. For all sets A and B, $\mathscr{P}(A \times B) = \mathscr{P}(A) \times \mathscr{P}(B)$.

18. Write a negation for each of the following statements. Indicate which is true, the statement or its negation. Justify your answers.
   a. $\forall$ sets S, $\exists$ a set T such that $S \cap T = \emptyset$.
   b. $\exists$ a set S such that $\forall$ sets T, $S \cup T = \emptyset$.

19. Let $S = \{a, b, c\}$ and for each integer $i = 0, 1, 2, 3$, let S_i be the set of all subsets of S that have i elements. List the elements in S_0, S_1, S_2, and S_3. Is $\{S_0, S_1, S_2, S_3\}$ a partition of $\mathscr{P}(S)$?

20. Let $S = \{a, b, c\}$ and let S_a be the set of all subsets of S that contain a, let S_b be the set of all subsets of S that contain b, let S_c be the set of all subsets of S that contain c, and let $S_\emptyset$ be the set whose only element is $\emptyset$. Is $\{S_a, S_b, S_c, S_\emptyset\}$ a partition of $\mathscr{P}(S)$?

21. Let $A = \{t, u, v, w\}$ and let S_1 be the set of all subsets of A that do not contain w and S_2 the set of all subsets of A that contain w.
   a. Find S_1.
   b. Find S_2.
   c. Are S_1 and S_2 disjoint?
   d. Compare the sizes of S_1 and S_2.
   e. How many elements are in $S_1 \cup S_2$?
   f. What is the relation between $S_1 \cup S_2$ and $\mathscr{P}(A)$?

H ∗**22.** The following problem, devised by Ginger Bolton, appeared in the January 1989 issue of the *College Mathematics Journal* (Vol. 20, No. 1, p. 68): Given a positive integer $n \geq 2$, let S be the set of all nonempty subsets of $\{2, 3, \ldots, n\}$. For each $S_i \in S$, let P_i be the product of the elements of S_i. Prove or disprove that

$$\sum_{i=1}^{2^{n-1}-1} P_i = \frac{(n+1)!}{2} - 1.$$

In 23 and 24 supply a reason for each step in the derivation.

23. For all sets A, B, and C,

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C).$$

Proof: Suppose A, B, and C are any sets. Then

$$\begin{aligned} (A \cup B) \cap C &= C \cap (A \cup B) && \text{by } \underline{\text{(a)}} \\ &= (C \cap A) \cup (C \cap B) && \text{by } \underline{\text{(b)}} \\ &= (A \cap C) \cup (B \cap C) && \text{by } \underline{\text{(c)}}. \end{aligned}$$

H **24.** For all sets A, B, and C, $(A \cup B) - (C - A) = A \cup (B - C)$.

Proof: Suppose A, B, and C are any sets. Then

$$\begin{aligned} (A \cup B) - (C - A) &= (A \cup B) \cap (C - A)^c && \text{by } \underline{\text{(a)}} \\ &= (A \cup B) \cap (C \cap A^c)^c && \text{by } \underline{\text{(b)}} \\ &= (A \cup B) \cap (A^c \cap C)^c && \text{by } \underline{\text{(c)}} \\ &= (A \cup B) \cap ((A^c)^c \cup C^c) && \text{by } \underline{\text{(d)}} \\ &= (A \cup B) \cap (A \cup C^c) && \text{by } \underline{\text{(e)}} \\ &= A \cup (B \cap C^c) && \text{by } \underline{\text{(f)}} \\ &= A \cup (B - C) && \text{by } \underline{\text{(g)}}. \end{aligned}$$

In 25–33 use the properties in Theorem 5.2.2 to construct an algebraic proof for the given statement.

25. For all sets A, B, and C,

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C).$$

26. For all sets A and B, $A \cup (B - A) = A \cup B$.

27. For all sets A, B, and C,

$$(A - B) - C = A - (B \cup C).$$

28. For all sets A and B, $A - (A - B) = A \cap B$.

29. For all sets A and B, $((A^c \cup B^c) - A)^c = A$.

30. For all sets A and B, $(B^c \cup (B^c - A))^c = B$.

31. For all sets A and B, $A - (A \cap B) = A - B$.

H **32.** For all sets A and B,

$$(A - B) \cup (B - A) = (A \cup B) - (A \cap B).$$

33. For all sets A, B, and C,

$$(A - B) - (B - C) = A - B.$$

In 34–36 use the properties in Theorem 5.2.2 to simplify the given expression.

H **34.** $A \cap ((B \cup A^c) \cap B^c)$

35. $(A - (A \cap B)) \cap (B - (A \cap B))$

36. $((A \cap (B \cup C)) \cap (A - B)) \cap (B \cup C^c)$

37. Consider the following set property: For all sets A and B, $A - B$ and B are disjoint.
 a. Use an element argument to derive the property.
 b. Use an algebraic argument to derive the property (by applying properties from Theorem 5.2.2).
 c. Comment on which method you found easier.

38. Consider the following set property: For all sets A, B, and C, $(A - B) \cup (B - C) = (A \cup B) - (B \cap C)$.
 a. Use an element argument to derive the property.
 b. Use an algebraic argument to derive the property (by applying properties from Theorem 5.2.2).
 c. Comment on which method you found easier.

Definition: Given sets A and B, the **symmetric difference of A and B,** denoted $A \triangle B$, is

$$A \triangle B = (A - B) \cup (B - A).$$

39. Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, and $C = \{5, 6, 7, 8\}$. Find each of the following sets:
 a. $A \triangle B$ b. $B \triangle C$
 c. $A \triangle C$ d. $(A \triangle B) \triangle C$

Refer to the definition of symmetric difference given above. Prove each of 40–45, assuming that A, B, and C are all subsets of a universal set U.

40. $A \triangle B = B \triangle A$ **41.** $A \triangle \emptyset = A$

42. $A \triangle A^c = U$ 43. $A \triangle A = \emptyset$

H **44.** If $A \triangle C = B \triangle C$, then $A = B$.

H **45.** $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

46. Derive the set identity $A \cup (A \cap B) = A$ from the properties listed in Theorem 5.2.2(1)–(9). Start by showing that for all subsets B of a universal set U, $U \cup B = U$. Then intersect both sides with A and deduce the identity.

47. Derive the set identity $A \cap (A \cup B) = A$ from the properties listed in Theorem 5.2.2(1)–(9). Start by showing that for all subsets B of a universal set U, $\emptyset = \emptyset \cap B$. Then take the union of both sides with A and deduce the identity.

In 48–50 assume that B is a Boolean algebra with operations $+$ and $\cdot$. Give the reasons needed to fill in the blanks in the proofs, but do not use any parts of Theorem 5.3.2 unless they have already been proved. You may use any part of the definition of a Boolean algebra and the results of previous exercises, however.

48. For all a in B, $a \cdot a = a$.

Proof: Let a be any element of B. Then

$$\begin{aligned} a &= a \cdot 1 && \underline{\text{(a)}} \\ &= a \cdot (a + \overline{a}) && \underline{\text{(b)}} \\ &= (a \cdot a) + (a \cdot \overline{a}) && \underline{\text{(c)}} \\ &= (a \cdot a) + 0 && \underline{\text{(d)}} \\ &= a \cdot a && \underline{\text{(e)}}. \end{aligned}$$

49. For all a in B, $a + 1 = 1$.

Proof: Let a be any element of B. Then

$$\begin{aligned} a + 1 &= a + (a + \overline{a}) && \underline{\text{(a)}} \\ &= (a + a) + \overline{a} && \underline{\text{(b)}} \\ &= a + \overline{a} && \text{by Example 5.3.6} \\ &= 1 && \underline{\text{(c)}}. \end{aligned}$$

50. For all a and b in B, $(a + b) \cdot a = a$.

Proof: Let a and b be any elements of B. Then

$$\begin{aligned} (a + b) \cdot a &= a \cdot (a + b) && \underline{\text{(a)}} \\ &= a \cdot a + a \cdot b && \underline{\text{(b)}} \\ &= a + a \cdot b && \underline{\text{(c)}} \\ &= a \cdot 1 + a \cdot b && \underline{\text{(d)}} \\ &= a \cdot (1 + b) && \underline{\text{(e)}} \\ &= a \cdot (b + 1) && \underline{\text{(f)}} \\ &= a \cdot 1 && \text{by exercise 49} \\ &= a && \underline{\text{(g)}}. \end{aligned}$$

In 51–57 assume that B is a Boolean algebra with operations $+$ and $\cdot$. Prove each statement without using any parts of Theorem 5.3.2 unless they have already been proved. You may use any part of the definition of a Boolean algebra and the results of previous exercises, however.

51. For all a in B, $a \cdot 0 = 0$.

52. For all a and b in B, $(a \cdot b) + a = a$.

53. $\overline{0} = 1$. 54. $\overline{1} = 0$

55. For all a and b in B, $\overline{a \cdot b} = \overline{a} + \overline{b}$. (*Hint:* Prove that $(a \cdot b) + (\overline{a} + \overline{b}) = 1$ and that $(a \cdot b) \cdot (\overline{a} + \overline{b}) = 0$, and use the fact that $a \cdot b$ has a unique complement.)

56. For all a and b in B, $\overline{a + b} = \overline{a} \cdot \overline{b}$.

H **57.** For all x, y, and z in B, if $x + y = x + z$ and $x \cdot y = x \cdot z$, then $y = z$.

58. Let $S = \{0, 1\}$, and define operations $+$ and $\cdot$ on S by the following tables:

+	0	1
0	0	1
1	1	1

·	0	1
0	0	0
1	0	1

 a. Show that the elements of S satisfy the following properties:
  (i) the commutative law for $+$
  (ii) the commutative law for $\cdot$
  (iii) the associative law for $+$
  (iv) the associative law for $\cdot$
  H **(v)** the distributive law for $+$ over $\cdot$
  (vi) the distributive law for $\cdot$ over $+$

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