Algorithm 4.1.1 Decimal to Binary Conversion Using Repeated Division by 2

[In Algorithm 4.1.1 the input is a nonnegative integer a. The aim of the algorithm is to produce a sequence of binary digits $r[0], r[1], r[2], \ldots, r[k]$ *so that the binary representation of a is*

$$(r[k]r[k-1]\cdots r[2]r[1]r[0])_2.$$

That is,

$$a = 2^k \cdot r[k] + 2^{k-1} \cdot r[k-1] + \cdots + 2^2 \cdot r[2] + 2^1 \cdot r[1] + 2^0 \cdot r[0].]$$

Input: a *[a nonnegative integer]*

Algorithm Body:

$q := a, i := 0$

[Repeatedly perform the integer division of q by 2 until q becomes 0. Store successive remainders in a one-dimensional array $r[0], r[1], r[2], \ldots, r[k]$*. Even if the initial value of q equals 0, the loop should execute one time (so that r[0] is computed). Thus the guard condition for the* **while** *loop is* $i = 0$ *or* $q \neq 0$.*]*

while ($i = 0$ or $q \neq 0$)

$r[i] := q \; mod \; 2$

$q := q \; div \; 2$

[r[i] and q can be obtained by calling the division algorithm.]

$i := i + 1$

end while

[After execution of this step, the values of $r[0], r[1], \ldots, r[i-1]$ *are all 0's and 1's, and* $a = (r[i-1]r[i-2]\cdots r[2]r[1]r[0])_2$.*]*

Output: $r[0], r[1], r[2], \ldots, r[i-1]$ *[a sequence of integers]*

Exercise Set 4.1*

Write the first four terms of the sequences defined by the formulas in 1–6.

1. $a_k = \dfrac{k}{10+k}$, for all integers $k \geq 1$.

2. $b_j = \dfrac{5-j}{5+j}$, for all integers $j \geq 1$.

3. $c_i = \dfrac{(-1)^i}{3^i}$, for all integers $i \geq 0$.

4. $d_m = 1 + \left(\dfrac{1}{2}\right)^m$ for all integers $m \geq 0$.

5. $e_n = \left\lfloor \dfrac{n}{2} \right\rfloor \cdot 2$, for all integers $n \geq 0$.

6. $f_n = \left\lfloor \dfrac{n}{4} \right\rfloor \cdot 4$, for all integers $n \geq 1$.

7. Let $a_k = 2k + 1$ and $b_k = (k-1)^3 + k + 2$ for all integers $k \geq 0$. Show that the first three terms of these sequences are identical but that their fourth terms differ.

Compute the first fifteen terms of each of the sequences in 8 and 9, and describe the general behavior of these sequences in words. (A definition of logarithm is given in Section 7.1.)

8. $g_n = \lfloor \log_2 n \rfloor$ for all integers $n \geq 1$.

9. $h_n = n \lfloor \log_2 n \rfloor$ for all integers $n \geq 1$.

Find explicit formulas for sequences of the form $a_1, a_2, a_3, \ldots$ with the initial terms given in 10–16.

10. $-1, 1, -1, 1, -1, 1$

11. $0, 1, -2, 3, -4, 5$

12. $\dfrac{1}{4}, \dfrac{2}{9}, \dfrac{3}{16}, \dfrac{4}{25}, \dfrac{5}{36}, \dfrac{6}{49}$

*For exercises with blue numbers or letters, solutions are given in Appendix B. The symbol ***H*** indicates that only a hint or a partial solution is given. The symbol ∗ signals that an exercise is more challenging than usual.

13. $1 - \frac{1}{2}, \frac{1}{2} - \frac{1}{3}, \frac{1}{3} - \frac{1}{4}, \frac{1}{4} - \frac{1}{5}, \frac{1}{5} - \frac{1}{6}, \frac{1}{6} - \frac{1}{7}$

14. $\frac{1}{3}, \frac{2}{9}, \frac{3}{27}, \frac{4}{81}, \frac{5}{243}, \frac{6}{729}$

15. $0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \frac{6}{7}$

16. 3, 6, 12, 24, 48, 96

✶ 17. Consider the sequence defined by $a_n = \frac{2n + (-1)^n - 1}{4}$ for all integers $n \geq 0$. Find an alternative explicit formula for a_n that uses the floor notation.

18. Let $a_0 = 2, a_1 = 3, a_2 = -2, a_3 = 1, a_4 = 0, a_5 = -1$, and $a_6 = -2$. Compute each of the summations and products below.

a. $\sum_{i=0}^{6} a_i$ **b.** $\sum_{i=0}^{0} a_i$ **c.** $\sum_{j=1}^{3} a_{2j}$ **d.** $\prod_{k=0}^{6} a_k$ e. $\prod_{k=2}^{2} a_k$

Compute the summations and products in 19–28.

19. $\sum_{k=1}^{5} (k+1)$ **20.** $\prod_{k=2}^{4} k^2$ 21. $\sum_{m=0}^{3} \frac{1}{2^m}$

22. $\prod_{j=0}^{4} (-1)^j$ **23.** $\sum_{i=1}^{1} i(i+1)$ 24. $\sum_{j=0}^{0} (j+1) \cdot 2^j$

25. $\prod_{k=2}^{2} \left(1 - \frac{1}{k}\right)$ 26. $\sum_{k=-1}^{1} (k^2 + 3)$

27. $\sum_{n=1}^{10} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ 28. $\prod_{i=2}^{5} \frac{i(i+2)}{(i-1) \cdot (i+1)}$

Write the summations in 29–31 in expanded form.

29. $\sum_{i=1}^{n} (-2)^i$ 30. $\sum_{j=1}^{n} j(j+1)$ 31. $\sum_{k=0}^{n} \frac{1}{k!}$

Write each of 32–41 using summation or product notation.

32. $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + 7^2$

33. $(1^3 - 1) - (2^3 - 1) + (3^3 - 1) - (4^3 - 1) + (5^3 - 1)$

34. $(2^2 - 1) \cdot (3^2 - 1) \cdot (4^2 - 1)$

35. $\frac{2}{3 \cdot 4} - \frac{3}{4 \cdot 5} + \frac{4}{5 \cdot 6} - \frac{5}{6 \cdot 7} + \frac{6}{7 \cdot 8}$

36. $1 - r + r^2 - r^3 + r^4 - r^5$

37. $(1 - t) \cdot (1 - t^2) \cdot (1 - t^3) \cdot (1 - t^4)$

38. $1^3 + 2^3 + 3^3 + \cdots + n^3$

39. $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!}$

40. $n + (n-1) + (n-2) + \cdots + 1$

41. $n + \frac{n-1}{2!} + \frac{n-2}{3!} + \frac{n-3}{4!} + \cdots + \frac{1}{n!}$

Compute each of 42–50.

42. $\frac{4!}{3!}$ 43. $\frac{6!}{8!}$ 44. $\frac{4!}{0!}$

45. $\frac{n!}{(n-1)!}$ **46.** $\frac{(n-1)!}{(n+1)!}$ 47. $\frac{n!}{(n-2)!}$

48. $\frac{((n+1)!)^2}{(n!)^2}$ **49.** $\frac{n!}{(n-k)!}$ 50. $\frac{n!}{(n-k+1)!}$

51. **a.** Prove that $n! + 2$ is divisible by 2, for all integers $n \geq 2$.
b. Prove that $n! + k$ is divisible by k, for all integers $n \geq 2$ and $k = 2, 3, \ldots, n$.
H **c.** Given any integer $m \geq 2$, is it possible to find a sequence of $m - 1$ consecutive positive integers none of which is prime? Explain your answer.

Transform each of 52 and 53 by making the change of variable $i = k + 1$.

52. $\sum_{k=0}^{5} k(k-1)$ 53. $\prod_{k=1}^{n} \frac{k}{k^2 + 4}$

Transform each of 54–57 by making the change of variable $j = i - 1$.

54. $\sum_{i=1}^{n+1} \frac{(i-1)^2}{i \cdot n}$ **55.** $\sum_{i=3}^{n} \frac{i}{i + n - 1}$

56. $\sum_{i=1}^{n-1} \frac{i}{(n-i)^2}$ 57. $\prod_{i=n}^{2n} \frac{n - i + 1}{n + i}$

Write each of 58–60 as a single summation or product.

58. $3 \cdot \sum_{k=1}^{n} (2k - 3) + \sum_{k=1}^{n} (4 - 5k)$

59. $2 \cdot \sum_{k=1}^{n} (3k^2 + 4) + 5 \cdot \sum_{k=1}^{n} (2k^2 - 1)$

60. $\left(\prod_{k=1}^{n} \frac{k}{k+1}\right) \cdot \left(\prod_{k=1}^{n} \frac{k+1}{k+2}\right)$

61. Check Theorem 4.1.1 for $m = 1$ and $n = 4$ by writing out the left-hand and right-hand sides of the equations in expanded form. The two sides are equal by repeated application of certain laws. What are these laws?

62. Suppose $a[1], a[2], a[3], \ldots, a[m]$ is a one-dimensional array and consider the following algorithm segment:

```
sum := 0
for k := 1 to m
    sum := sum + a[k]
next k
```

Fill in the blanks below so that each algorithm segment performs the same job as the one given above.

a.
```
sum := 0
for i := 0 to ______
    sum := ______
next i
```

b.
```
sum := 0
for j := 2 to ______
    sum := ______
next j
```

Exercise Set 4.2

1. Use mathematical induction (and the proof of Proposition 4.2.1 as a model) to show that any amount of money of at least 14¢ can be made up using 3¢ and 8¢ coins.

2. Use mathematical induction to show that any postage of at least 12¢ can be obtained using 3¢ and 7¢ stamps.

3. For each positive integer n, let $P(n)$ be the formula

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

a. Write $P(1)$. Is $P(1)$ true?
b. Write $P(k)$.
c. Write $P(k+1)$.
d. In a proof by mathematical induction that the formula holds for all integers $n \geq 1$, what must be shown in the inductive step?

4. For each integer n with $n \geq 2$, let $P(n)$ be the formula

$$\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}.$$

a. Write $P(2)$. Is $P(2)$ true?
b. Write $P(k)$.
c. Write $P(k+1)$.
d. In a proof by mathematical induction that the formula holds for all integers $n \geq 2$, what must be shown in the inductive step?

5. Fill in the missing pieces in the following proof that

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

for all integers $n \geq 1$.

Proof: Let the property $P(n)$ be the equation

$$1 + 3 + 5 + \cdots + (2n-1) = n^2.$$

Show that the property is true for n = 1: To establish the property for $n = 1$, we must show that when 1 is substituted in place of n, the left-hand side equals the right-hand side. But when $n = 1$, the left-hand side is the sum of all the odd integers from 1 to $2 \cdot 1 - 1$, which is the sum of the odd integers from 1 to 1, which is just 1. The right-hand side is _(a)_, which also equals 1. So the property is true for $n = 1$.

Show that for all integers k ≥ 1, if the property is true for n = k then it is true for n = k + 1: Let k be any integer with $k \geq 1$.

[*Suppose the property* $1 + 3 + 5 + \cdots + (2n-1) = n^2$ *is true when k is substituted for n.*]

Suppose $1 + 3 + 5 + \cdots + (2k-1) =$ _(b)_.

[*This is the inductive hypothesis.*]

[*We must show that the property is true when k + 1 is substituted for n*].

We must show that

_(c)_ = _(d)_. 4.2.5

But the left-hand side of equation (4.2.5) is

$1 + 3 + 5 + \cdots + (2(k+1) - 1)$

$= 1 + 3 + 5 + \cdots + (2k+1)$ by algebra

$= [1 + 3 + 5 + \cdots + (2k-1)] + (2k+1)$

the next-to-last term is $2k - 1$ because _(e)_

$= k^2 + (2k+1)$ by _(f)_

$= (k+1)^2$ by algebra

which is the right-hand side of equation (4.2.5) [*as was to be shown*].

[*Since we have proved the basis step and the inductive step, we conclude that the given statement is true.*]

The proof above was heavily annotated to help make its logical flow more obvious. In standard mathematical writing, such annotation is omitted.

Prove each statement in 6–9 using mathematical induction. Do not derive them from Theorem 4.2.2 or Theorem 4.2.3.

6. For all integers $n \geq 1$, $2 + 4 + 6 + \cdots + 2n = n^2 + n$.

7. For all integers $n \geq 1$,

$$1 + 6 + 11 + 16 + \cdots + (5n-4) = \frac{n(5n-3)}{2}.$$

8. For all integers $n \geq 0$, $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$.

9. For all integers $n \geq 3$,

$$4^3 + 4^4 + 4^5 + \cdots + 4^n = \frac{4(4^n - 16)}{3}.$$

Prove each of the statements in 10–17 by mathematical induction.

10. $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$, for all integers $n \geq 1$.

11. $1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$, for all integers $n \geq 1$.

12. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$, for all integers $n \geq 1$.

13. $\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}$, for all integers $n \geq 2$.

14. $\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$, for all integers $n \geq 0$.

H **15.** $\sum_{i=1}^{n} i(i!) = (n+1)! - 1$, for all integers $n \geq 1$.

16. $\left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$, for all integers $n \geq 2$.

17. $\prod_{i=0}^{n}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)=\frac{1}{(2n+2)!}$, for all integers $n \geq 0$.

H ✶**18.** If x is a real number not divisible by π, then for all integers $n \geq 1$,

$$\sin x + \sin 3x + \sin 5x + \cdots + \sin(2n-1)x = \frac{1-\cos 2nx}{2\sin x}.$$

Use the formula for the sum of the first n integers and/or the formula for the sum of a geometric sequence to find the sums in 19–28.

19. $4+8+12+16+\cdots+200$

20. $5+10+15+20+\cdots+300$

21. $3+4+5+6+\cdots+1000$

22. $7+8+9+10+\cdots+600$

23. $1+2+3+\cdots+(k-1)$, where k is an integer and $k \geq 2$

24. a. $1+2+2^2+\cdots+2^{25}$
b. $2+2^2+2^3+\cdots+2^{26}$

25. $3+3^2+3^3+\cdots+3^n$, where n is an integer with $n \geq 1$

26. $5^3+5^4+5^5+\cdots+5^k$, where k is any integer with $k \geq 3$.

27. $1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}$, where n is a positive integer

28. $1-2+2^2-2^3+\cdots+(-1)^n 2^n$, where n is a positive integer

H **29.** Find a formula in n, a, m, and d for the sum $(a+md)+(a+(m+1)d)+(a+(m+2)d)+\cdots+(a+(m+n)d)$, where m and n are integers, $n \geq 0$, and a and d are real numbers. Justify your answer.

30. Find a formula in a, r, m, and n for the sum $ar^m + ar^{m+1} + ar^{m+2} + \cdots + ar^{m+n}$, where m and n are integers, $n \geq 0$, and a and r are real numbers. Justify your answer.

31. You have two parents, four grandparents, eight great-grandparents, and so forth.
   a. If all your ancestors were distinct, what would be the total number of your ancestors for the past 40 generations (counting your parents' generation as number one)? (*Hint:* Use the formula for the sum of a geometric sequence.)
   b. Assuming that each generation represents 25 years, how long is 40 generations?
   c. The total number of people who have ever lived is approximately 10 billion, which equals 10^{10} people. Compare this fact with the answer to part (a). What do you deduce?

32. Find the mistake in the following proof fragment.

Theorem: For any integer $n \geq 1$,

$$1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}.$$

"**Proof (by mathematical induction):** Certainly the theorem is true for $n=1$ because $1^2=1$ and $\frac{1(1+1)(2\cdot 1+1)}{6}=1$. So the basis step is true.

For the inductive step, suppose that for some integer $k \geq 1$, $k^2=\frac{k(k+1)(2k+1)}{6}$. We must show that $(k+1)^2=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}\ldots$."

✶33. Use Theorem 4.2.2 to prove that if m and n are any positive integers and m is odd, then $\sum_{k=0}^{m-1}(n+k)$ is divisible by m. Does the conclusion hold if m is even? Justify your answer.

H ✶**34.** Use Theorem 4.2.2 and the result of exercise 10 to prove that if p is any prime number with $p \geq 5$, then the sum of squares of any p consecutive integers is divisible by p.

4.3 Mathematical Induction II

A good proof is one which makes us wiser. —I. Manin, *A Course in Mathematical Logic,* 1977

In natural science courses, deduction and induction are presented as alternative modes of thought—deduction being to infer a conclusion from general principles using the laws of logical reasoning, and induction being to enunciate a general principle after observing it to hold in a large number of specific instances. In this sense, then, *mathematical* induction is not inductive but deductive. Once proved by mathematical induction, a theorem is known just as certainly as if it were proved by any other mathematical method. Inductive reasoning, in the natural sciences sense, *is* used in mathematics, but only to

We must show that $a_{k+1} = 2 \cdot 5^{(k+1)-1} = 2 \cdot 5^k$.

But the left-hand side of the equation is

$$\begin{aligned} a_{k+1} &= 5a_{(k+1)-1} && \text{by definition of } a_1, a_2, a_3, \ldots \\ &= 5a_k && \text{since } (k+1)-1 = k \\ &= 5 \cdot (2 \cdot 5^{k-1}) && \text{by inductive hypothesis} \\ &= 2 \cdot (5 \cdot 5^{k-1}) && \text{by regrouping} \\ &= 2 \cdot 5^k && \text{by the laws of exponents} \end{aligned}$$

which is the right-hand side of the equation [*as was to be shown*].

[*Since we have proved the basis step and the inductive step, we conclude that the formula holds for all terms of the sequence.*] ■

Exercise Set 4.3

1. Based on the discussion of the product $(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{4})\cdots(1-\frac{1}{n})$ at the beginning of this section, conjecture a formula for general n. Prove your conjecture by mathematical induction.

2. Experiment with computing values of the product $(1+\frac{1}{1})(1+\frac{1}{2})(1+\frac{1}{3})\cdots(1+\frac{1}{n})$ for small values of n to conjecture a formula for this product for general n. Prove your conjecture by mathematical induction.

3. Observe that

$$\frac{1}{1\cdot 3} = \frac{1}{3}$$
$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} = \frac{2}{5}$$
$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} = \frac{3}{7}$$
$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \frac{1}{7\cdot 9} = \frac{4}{9}$$

Guess a general formula and prove it by mathematical induction.

H **4.** Observe that

$$1 = 1,$$
$$1 - 4 = -(1+2),$$
$$1 - 4 + 9 = 1 + 2 + 3,$$
$$1 - 4 + 9 - 16 = -(1+2+3+4),$$
$$1 - 4 + 9 - 16 + 25 = 1 + 2 + 3 + 4 + 5.$$

Guess a general formula and prove it by mathematical induction.

5. Evaluate the sum $\sum_{k=1}^{n} \frac{k}{(k+1)!}$ for $n = 1, 2, 3, 4,$ and 5. Make a conjecture about a formula for this sum for general n, and prove your conjecture by mathematical induction.

6. For each positive integer n, let $P(n)$ be the property

$$5^n - 1 \text{ is divisible by } 4.$$

a. Write $P(0)$. Is $P(0)$ true?
b. Write $P(k)$.
c. Write $P(k+1)$.
d. In a proof by mathematical induction that this divisibility property holds for all integers $n \geq 0$, what must be shown in the inductive step?

7. For each positive integer n, let $P(n)$ be the property

$$2^n < (n+1)!.$$

a. Write $P(2)$. Is $P(2)$ true?
b. Write $P(k)$.
c. Write $P(k+1)$.
d. In a proof by mathematical induction that this inequality holds for all integers $n \geq 2$, what must be shown in the inductive step?

Prove each statement in 8–23 by mathematical induction.

8. $5^n - 1$ is divisible by 4, for each integer $n \geq 0$.

9. $7^n - 1$ is divisible by 6, for each integer $n \geq 0$.

10. $n^3 - 7n + 3$ is divisible by 3, for each integer $n \geq 0$.

11. $3^{2n} - 1$ is divisible by 8, for each integer $n \geq 0$.

12. For any integer $n \geq 1$, $7^n - 2^n$ is divisible by 5.

H **13.** For any integer $n \geq 1$, $x^n - y^n$ is divisible by $x - y$, where x and y are any integers with $x \neq y$.

H **14.** $n^3 - n$ is divisible by 6, for each integer $n \geq 2$.

15. $n(n^2+5)$ is divisible by 6, for each integer $n \geq 1$.

16. $2^n < (n+1)!$, for all integers $n \geq 2$.

17. $1 + 3n \leq 4^n$, for every integer $n \geq 0$.

18. $5^n + 9 < 6^n$, for all integers $n \geq 2$.

19. $n^2 < 2^n$, for all integers $n \geq 5$.

20. $2^n < (n+2)!$, for all integers $n \geq 0$.

21. $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$, for all integers $n \geq 2$.

22. $1 + nx \leq (1+x)^n$, for all real numbers $x > -1$ and integers $n \geq 2$.

23. a. $n^3 > 2n + 1$, for all integers $n \geq 2$.
b. $n! > n^2$, for all integers $n \geq 4$.

24. A sequence $a_1, a_2, a_3, \ldots$ is defined by letting $a_1 = 3$ and $a_k = 7a_{k-1}$ for all integers $k \geq 2$. Show that $a_n = 3 \cdot 7^{n-1}$ for all integers $n \geq 1$.

25. A sequence $b_0, b_1, b_2, \ldots$ is defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$ for all integers $k \geq 1$. Show that $b_n > 4n$ for all integers $n \geq 0$.

26. A sequence $c_0, c_1, c_2, \ldots$ is defined by letting $c_0 = 3$ and $c_k = (c_{k-1})^2$ for all integers $k \geq 1$. Show that $c_n = 3^{2^n}$ for all integers $n \geq 0$.

27. A sequence $d_1, d_2, d_3, \ldots$ is defined by letting $d_1 = 2$ and $d_k = \frac{d_{k-1}}{k}$ for all integers $k \geq 2$. Show that for all integers $n \geq 1$, $d_n = \frac{2}{n!}$.

28. Prove that for all integers $n \geq 1$,

$$\frac{1}{3} = \frac{1+3}{5+7} = \frac{1+3+5}{7+9+11} = \cdots$$
$$= \frac{1+3+\cdots+(2n-1)}{(2n+1)+\cdots+(4n-1)}.$$

29. As each of a group of business people arrives at a meeting, each shakes hands with all the other people present. Use mathematical induction to show that if n people come to the meeting then $[n(n-1)]/2$ handshakes occur.

In order for a proof by mathematical induction to be valid, the basis statement must be true for $n = a$ and the argument of the inductive step must be correct for every integer $k \geq a$. In 30 and 31 find the mistakes in the "proofs" by mathematical induction.

30. **"Theorem:"** For any integer $n \geq 1$, all the numbers in a set of n numbers are equal to each other.

"Proof (by mathematical induction): It is obviously true that all the numbers in a set consisting of just one number are equal to each other, so the basis step is true. For the inductive step, let $A = \{a_1, a_2, \ldots, a_k, a_{k+1}\}$ be any set of $k+1$ numbers. Form two subsets each of size k:

$$B = \{a_1, a_2, a_3, \ldots, a_k\} \quad \text{and}$$
$$C = \{a_1, a_3, a_4, \ldots, a_{k+1}\}.$$

(B consists of all the numbers in A except a_{k+1}, and C consists of all the numbers in A except a_2.) By inductive hypothesis, all the numbers in B equal a_1 and all the numbers in C equal a_1 (since both sets have only k numbers). But every number in A is in B or C, so all the numbers in A equal a_1; hence all are equal to each other."

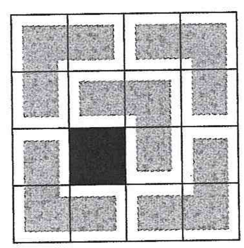
31. **"Theorem:"** For all integers $n \geq 1$, $3^n - 2$ is even.

"Proof (by mathematical induction): Suppose the theorem is true for an integer k, where $k \geq 1$. That is, suppose that $3^k - 2$ is even. We must show that $3^{k+1} - 2$ is even. But

$$3^{k+1} - 2 = 3^k \cdot 3 - 2 = 3^k(1+2) - 2$$
$$= (3^k - 2) + 3^k \cdot 2.$$

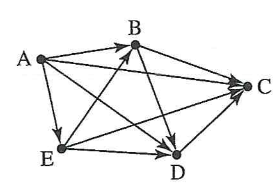
Now $3^k - 2$ is even by inductive hypothesis and $3^k \cdot 2$ is even by inspection. Hence the sum of the two quantities is even (by Theorem 3.1.1). It follows that $3^{k+1} - 2$ is even, which is what we needed to show."

32. An L-tromino, or tromino for short, is similar to a domino but is shaped like an L: ▙. Call a checkerboard that is formed using m squares on a side an $m \times m$ checkerboard. If one square is removed from a 4×4 checkerboard, the remaining squares can be completely covered by trominos. For instance, a covering for one such board is the following:



Use mathematical induction to prove that for any integer $n \geq 1$, if one square is removed from a $2^n \times 2^n$ checkerboard, the remaining squares can be completely covered by trominos.

33. In a round-robin tournament each team plays every other team exactly once. If the teams are labeled $T_1, T_2, \ldots, T_n$, then the outcome of such a tournament can be represented by a drawing, called a *directed graph,* in which the teams are represented as dots and an arrow is drawn from one dot to another if, and only if, the team represented by the first dot beats the team represented by the second dot. For example, the directed graph below shows one outcome of a round-robin tournament involving five teams, A, B, C, D, and E.



Use mathematical induction to show that in any round-robin tournament involving n teams, where $n \geq 2$, it is possible to

Furthermore, $r < d$. *[For suppose $r \geq d$. Then*

$$n - d(q+1) = n - dq - d = r - d \geq 0,$$

and so $n - d(q+1)$ would be a nonnegative integer in S that would be smaller than r. But r is the smallest integer in S. This contradiction shows that the supposition $r \geq d$ must be false.] The preceding arguments prove that there exist integers r and q for which

$$n = dq + r \quad \text{and} \quad 0 \leq r < d.$$

[This is what was to be shown.]

Another consequence of the well-ordering principle is the fact that any strictly decreasing sequence of nonnegative integers is finite. That is, if $r_1, r_2, r_3, \ldots$ is a sequence of nonnegative integers satisfying

$$r_i > r_{i+1}$$

for all $i \geq 1$, then $r_1, r_2, r_3, \ldots$ is a finite sequence. *[For by the well-ordering principle such a sequence would have to have a least element r_k. It follows that r_k must be the final term of the sequence because if there were a term r_{k+1}, then since the sequence is strictly decreasing, $r_{k+1} < r_k$, which would be a contradiction.]* This fact is frequently used in computer science to prove that algorithms terminate after a finite number of steps and to prove that the guard conditions for loops eventually become false. It was also used implicitly in the proof of Theorem 3.3.2 and to justify the claim in Section 3.8 that the Euclidean algorithm eventually terminates.

Exercise Set 4.4

1. Suppose $a_1, a_2, a_3, \ldots$ is a sequence defined as follows:

$$a_1 = 1, a_2 = 3,$$
$$a_k = a_{k-2} + 2a_{k-1} \quad \text{for all integers } k \geq 3.$$

Prove that a_n is odd for all integers $n \geq 1$.

2. Suppose $b_1, b_2, b_3, \ldots$ is a sequence defined as follows:

$$b_1 = 4, b_2 = 12$$
$$b_k = b_{k-2} + b_{k-1} \quad \text{for all integers } k \geq 3.$$

Prove that b_n is divisible by 4 for all integers $n \geq 1$.

3. Suppose that $c_0, c_1, c_2, \ldots$ is a sequence defined as follows:

$$c_0 = 2, c_1 = 2, c_2 = 6,$$
$$c_k = 3c_{k-3} \quad \text{for all integers } k \geq 3.$$

Prove that c_n is even for all integers $n \geq 0$.

4. Suppose that $d_1, d_2, d_3, \ldots$ is a sequence defined as follows:

$$d_1 = \frac{9}{10}, d_2 = \frac{10}{11},$$
$$d_k = d_{k-1} \cdot d_{k-2} \quad \text{for all integers } k \geq 3.$$

Prove that $0 < d_n \leq 1$ for all integers $n \geq 0$.

5. Suppose that $e_0, e_1, e_2, \ldots$ is a sequence defined as follows:

$$e_0 = 1, e_1 = 2, e_2 = 3,$$
$$e_k = e_{k-1} + e_{k-2} + e_{k-3} \quad \text{for all integers } k \geq 3.$$

Prove that $e_n \leq 3^n$ for all integers $n \geq 0$.

6. Suppose that $f_1, f_2, f_3, \ldots$ is a sequence defined as follows:

$$f_1 = 1, f_k = 2 \cdot f_{\lfloor k/2 \rfloor} \quad \text{for all integers } k \geq 2.$$

Prove that $f_n \leq n$ for all integers $n \geq 1$.

7. Suppose that $g_0, g_1, g_2, \ldots$ is a sequence defined as follows:

$$g_0 = 12, g_1 = 29,$$
$$g_k = 5g_{k-1} - 6g_{k-2} \quad \text{for all integers } k \geq 2.$$

Prove that $g_n = 5 \cdot 3^n + 7 \cdot 2^n$ for all integers $n \geq 0$.

8. Suppose that $h_0, h_1, h_2, \ldots$ is a sequence defined as follows:

$$h_0 = 1, h_1 = 2, h_2 = 3,$$
$$h_k = h_{k-1} + h_{k-2} + h_{k-3} \quad \text{for all integers } k \geq 3.$$

a. Prove that $h_n \leq 3^n$ for all integers $n \geq 0$.

b. Suppose that s is any real number such that $s^3 \geq s^2 + s + 1$. (This implies that $s > 1.83$.) Prove that $h_n \leq s^n$ for all $n \geq 2$.

9. Define a sequence $a_1, a_2, a_3, \ldots$ as follows: $a_1 = 1, a_2 = 3$, and $a_k = a_{k-1} + a_{k-2}$ for all integers $k \geq 3$. (This sequence is known as the Lucas sequence.) Use strong mathematical induction to prove that $a_n \leq \left(\frac{7}{4}\right)^n$ for all integers $n \geq 1$.

10. You begin solving a jigsaw puzzle by finding two pieces that match and fitting them together. Each subsequent step of the solution consists of fitting together two blocks made up of one or more pieces that have previously been assembled. Use strong mathematical induction to prove that the number of steps required to put together all n pieces of a jigsaw puzzle is $n - 1$.

***H* 11.** Use strong mathematical induction to prove the existence part of the unique factorization theorem: Every integer greater than or equal to 2 is either a prime number or a product of prime numbers.

12. Any product of two or more integers is a result of successive multiplications of two integers at a time. For instance, here are a few of the ways in which $a_1a_2a_3a_4$ might be computed: $(a_1a_2)(a_3a_4)$ or $((a_1a_2)a_3)a_4)$ or $a_1((a_2a_3)a_4)$. Use strong mathematical induction to prove that any product of two or more odd integers is odd.

13. Any sum of two or more integers is a result of successive additions of two integers at a time. For instance, here are a few of the ways in which $a_1 + a_2 + a_3 + a_4$ might be computed: $(a_1 + a_2) + (a_3 + a_4)$ or $((a_1 + a_2) + a_3) + a_4)$ or $a_1 + ((a_2 + a_3) + a_4)$. Use strong mathematical induction to prove that any sum of two or more even integers is even.

***H* 14.** Use strong mathematical induction to prove that for any integer $n \geq 2$, if n is even, then any sum of n odd integers is even, and if n is odd, then any sum of n odd integers is odd.

15. Compute $4^1, 4^2, 4^3, 4^4, 4^5, 4^6, 4^7$, and 4^8. Make a conjecture about the units digit of 4^n where n is a positive integer. Use strong mathematical induction to prove your conjecture.

16. Compute $3^0, 3^1, 3^2, 3^3, 3^4, 3^5, 3^6, 3^7, 3^8, 3^9$, and 3^{10}. Make a conjecture about the units digit of 3^n where n is a positive integer. Use strong mathematical induction to prove your conjecture.

17. Find the mistake in the following "proof" that purports to show that every nonnegative integer power of every nonzero real number is 1.

"**Proof:** Let r be any nonzero real number and let the property $P(n)$ be the equation "$r^n = 1$."

Show that the property is true for n = 0: The property is true for $n = 0$ because $r^0 = 1$ by definition of zeroth power.

Show that for all integers k > 0, if the property is true for all integers i with 0 ≤ i < k, then it is true for k: Let $k > 0$ be an integer, and suppose that $r^i = 1$ for all integers i with $0 \leq i < k$. [*We must show that* $r^k = 1$.] Now

r^k	$= r^{(k-1)+(k-1)-(k-2)}$	because $(k-1) + (k-1) - (k-2) = k$
	$= \dfrac{r^{k-1} \cdot r^{k-1}}{r^{k-2}}$	by the laws of exponents
	$= \dfrac{1 \cdot 1}{1}$	by inductive hypothesis
	$= 1.$	

Thus $r^k = 1$ [*as was to be shown*].

[*Since we have proved the basis step and the inductive step, we conclude that $r^n = 1$ for all integers $n \geq 0$.*]"

18. Use the well-ordering principle to prove Theorem 3.3.2: Every integer greater than 1 is divisible by a prime number.

19. Use the well-ordering principle to prove that every integer n greater than 1 is either a prime number or a product of prime numbers.

20. The Archimedean property for the rational numbers states that for all rational numbers r, there is an integer n such that $n > r$. Prove this property.

***H* 21.** Use the result of exercise 20 and the well-ordering principle for the integers to show that given any rational number r, there is an integer m such that $m \leq r < m + 1$.

22. Use the well-ordering principle to prove that given any integer $n \geq 1$, there exists an odd integer m and a nonnegative integer k such that $n = 2^k \cdot m$.

✶ 23. Use the well-ordering principle to prove that if a and b are any integers not both zero, then there exist integers u and v such that $\gcd(a, b) = ua + vb$. (*Hint:* Let S be the set of all positive integers of the form $ua + vb$ for some integers u and v.)

24. Suppose $P(n)$ is a property such that
1. $P(0), P(1), P(2)$ are all true,
2. for all integers $k \geq 0$, if $P(k)$ is true, then $P(3k)$ is true.

Must it follow that $P(n)$ is true for all integers $n \geq 0$? If yes, explain why; if no, give a counterexample.

25. Prove that if a statement can be proved by strong mathematical induction, then it can be proved by ordinary mathematical induction. To do this, let $P(n)$ be a property that is defined for integers n, and suppose the following two statements are true:
1. $P(a), P(a+1), P(a+2), \ldots, P(b)$.
2. For any integer $k > b$, if $P(i)$ is true for all integers i with $a \leq i < k$, then $P(k)$ is true.

The principle of strong mathematical induction would allow us to conclude immediately that $P(n)$ is true for all integers $n \geq a$. Can we reach the same conclusion using the principle of ordinary mathematical induction? Yes! To see